Finite Elasticity Theory

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PREFACE

This book is based on notes developed for a one-semester course offered at Berkeley. Typically, this serves graduate engineering students studying Mechanics, but also occasionally attracts interest on the part of students studying Mathematics and Physics. For this reason, and to suit my own predilections, the level of mathematical rigor is appropriate for readers possessing a relatively modest background. This has the pedagogical advantage of allowing time to make contact with physical phenomena, while providing context for such mathematical concepts as are needed to support their modeling and analysis. Advanced readers seeking more than this should consult the books by Antman (2005), Ciarlet (1998), and Šilhavý (1997), for example. My expectation, and part of the motivation for this work, is that books and treatises of the latter kind may be more fully appreciated by students after reading an introductory course.

Throughout the book, we focus on the purely mechanical theory. However, extensive reference will be made of the notions of work, energy, and, in the final chapter, dissipation.

The emphasis here is on developing a framework for the phenomenological theory. Despite what contemporary students are often taught, such theories remain the best hope for the quantitative study of physical phenomena occurring on human (macroscopic) scales of length and time. This is perhaps best illustrated by our own subject, which developed rapidly after the introduction of a clear and concise framework for phenomenological modeling. Thus, researchers began to exploit the predictive potential of the theory of nonlinear elasticity only after constitutive relations derived from statistical mechanics were largely abandoned in favor of those of phenomenological origin, which could be fitted to actual data. In turn, nonlinear elasticity, because of its secure logical, physical, and mathematical foundations, has served as a template for the development of theories of inelasticity, continuum electrodynamics, structural mechanics, thermodynamics, diffusion, rheology, biophysics, growth mechanics, and so on. The final chapter, consisting of a brief introduction to plasticity theory, illustrates how elasticity interacts with and informs other branches of solid mechanics. In short, the study of nonlinear elasticity is fundamental to the understanding of those aspects of modern mechanics research that are of greatest interest and relevance.

These notes are mainly about the conceptual foundations of nonlinear elasticity and the formulation of problems, occasionally including a worked-out solution. The latter are quite rare, due the nonlinearity of the equations to be solved, and so recourse must usually be made to numerical methods, which, however, lie outside the scope of this book. Explicit solutions are of great importance, however, because they offer a means of establishing a direct correlation between theory and experiment, and thus extracting definitive information about the constitutive equations underpinning the theory for use in computations.

Although elasticity theory is inherently nonlinear, courses on the purely linear theory, treating the equations obtained by formally linearizing the general theory, are quite prevalent. This is due to the great utility of the linear theory in solving problems that arise in engineering and physics. To a large degree, and mainly for historical reasons, such courses are delivered independently of courses of the present kind. The explanation for this schism is that the nonlinear theory did not come into its own until the latter half of the last century and, by then, the linear theory had matured into a major discipline in its own right, on par with classical fluid dynamics, heat transfer, and other branches of the applied and engineering sciences. This fueled research on applications of the theory relying on and, in turn, advancing techniques for treating elliptic linear partial differential equations. The word *Finite* in the title refers to the possibly large deformations covered by the nonlinear theory, as distinct from the infinitesimal deformations to which the linear theory is limited. Elasticity theory is, nevertheless, *non*linear and the use of the linear approximation to it should always be justified, in the circumstances at hand, by checking its predictions against the assumptions made in the course of obtaining the equations. However, this is inconvenient and, thus, almost never done in practice.

Unfortunately, all this is somewhat disquieting from the standpoint of contemporary students, who must grapple not only with the question of whether or not a problem may be modeled using elasticity theory, but may also feel obliged to categorize it as either linear or nonlinear at the outset. Those more interested in concepts and in the formulation of new theories of the kind mentioned above will derive much value from an understanding of nonlinear elasticity, whereas my view is that linear elasticity has virtually nothing to offer in this regard, due to the severe restrictions underpinning its foundations.

The book collects what I think students should know about the subject before embarking on research, including my interpretations of modern works that have aided me in refining my own understanding. Those seeking to grasp how and why materials work the way they do may be disappointed. For them I recommend Gordon (1968, 1978) as an engaging source of knowledge that should ideally be acquired, but which rarely, if ever, is, before reading any textbook on the mechanics of materials. In particular, these may be read in lieu of an undergraduate course on Strength of Materials, which is to be avoided at all costs. If the present book comes to be regarded as a worthy supplement to, say, Ogden's modern classic *Nonlinear Elastic Deformations* (1997), then I will regard the writing of it to have been worthwhile. Readers having a grasp of continuum mechanics, say at the level of Chadwick's pocketbook *Continuum Mechanics: Concise Theory and Problems* (1976), will have no trouble getting started. Reference should be made to that excellent text for any concept encountered here that may be unfamiliar. The reader is cautioned that current fashion in continuum mechanics is to rely largely on direct notation. Indeed, while this invariably serves the interests of clarity when discussing the conceptual foundations of the subject, there are circumstances that call for the use of Cartesian index notation, and we shall avail ourselves of it when doing so proves to be helpful. We adopt the usual summation convention for repeated subscripts together with the rule that subscripts preceded by commas always indicate partial differentiation with respect to the Cartesian coordinates. Direct notation is really only useful to the extent that it so closely resembles Cartesian index notation, while the latter, being operational in nature, is invariably the setting of choice for carrying out the more involved calculations.

Some topics are given more attention than others, in accordance with my personal views about their relative importance and the extent to which they are adequately covered, or more often not, in the textbook and monograph literatures. My intention to use these notes in my future teaching of the material leads me to buck the current trend and *not* include answers to the exercises. The latter are sprinkled throughout the text, and an honest attempt to solve them constitutes an integral part of the course. The book is definitely *not* self-contained. Readers are presumed to have been exposed to a first course on continuum mechanics, and the standard results that are always taught in such a course are frequently invoked without derivation. In particular, readers are expected to have a working knowledge of tensor analysis in Euclidean three-space and the reason why tensors are used in the formulation of physical theories—roughly, to ensure that the predictions of such theories are not dependent on the manner in which we coordinatize space for our own convenience.

The contemporary books by Liu (2002), and by Gurtin, Fried and Anand (2010) can be heartily recommended as a point of departure for those wishing to understand the foundations for modern applications of continuum mechanics. A vast amount of important material is also contained in Truesdell and Noll's *Nonlinear Field Theories of Mechanics* (1965) and Rivlin's *Collected Works* (Barenblatt and Joseph, 1997), which should be read by anyone seeking a firm understanding of nonlinear elasticity and continuum theory in general.

David Steigmann, Berkeley 2016

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Concept of an elastic material

One would think this would be the easiest chapter to write, but alas such is not the case. Thus, we will have to settle for the present, rather superficial substitute, which may be skipped over by anyone—and thus presumably everyone reading this book—who has some passing acquaintance with the concept of elasticity. When attempting to define the property we call *elasticity*, and how to recognize it when we see it, we encounter certain non-trivial obstacles, not least among these being the fact that elastic materials per se do not actually exist. That is, there are no known examples of materials whose responses to stimuli conform to conventional notions of elasticity in all circumstances. In fact, even the concept of ideal elastic response is open to a wide range of interpretations. Rather than delve into the underlying philosophical questions, for which I am not qualified, I defer to the thought-provoking account contained in a contemporary article by Rajagopal (2011).

For our purposes, the idea of elasticity may by abstracted from the simplest observations concerning the extension of a rubber band, say, to a certain length. Naturally, one finds that a force is required to do so and, if the band is left alone for a period of time, that this force typically settles to a more-or-less fixed value that depends on the length. This is not to say that the force remains at that value indefinitely, but often there is a substantial interval of time, encompassing the typical human attention span, during which it does. More often, one fixes the force, f say, by hanging a weight of known amount from one end; the length of the band adjusts accordingly, reaching a corresponding value that is sensibly fixed over some time interval. If one has a graph of force vs. length, then usually one can read off the force corresponding to a given length and vice versa. The situation for a typical rubber band is shown in Figure 1.1, where the abscissa is scaled by the original (unforced) length of the band. This scaling, denoted by λ, is called the *stretch*.

If one looks closely one may observe a slight hysteresis on this graph. This is due to small-scale defects or irregularities among the long-chain molecules of which the rubber is made. They have the effect of impeding attainment of the optimal or energy minimizing state of the material under load, and are usually reduced to the point of being negligible by subjecting the rubber to a cyclic strain, which effectively "works the kinks out." This is known as the Mullins effect. Studies of it in the mechanics literature are confined mostly to its description and prediction, based on phenomenological theory (see Ogden's paper, 2004), rather than its explanation. A notable exception is the book by Müller and Strehlow (2010),

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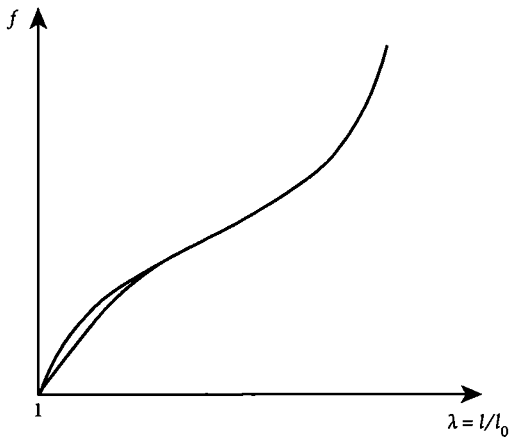

Figure 1.1 Uniaxial force-extension relation for rubber. Stiffening is due to straightening of long-chain molecules

which offers an interesting explanation in terms of microstructural instabilities and associated thermodynamics. For the most part these subjects will not be covered, although we will devote considerable attention later to the notion of stability and its connection to energy minimization.

Ignoring hysteresis, then, we can expect to extract a relation of the form $f = F(\lambda)$ from a graph of the data. Here, F is a *constitutive* function; i.e., a function that codifies the nature of the material in terms of its response to deformation. We are justified in attributing the function to material properties—and not just the nature of the experiment—provided that the material is uniform and no other forces are acting. In this case, equilibrium considerations yield the conclusion that the forces acting at the ends of an arbitrary segment of the band are opposed in direction, but equal in magnitude, the common value of the latter being given by the force f. If the stretch, which is really a function defined pointwise, is also uniform, then it can be correlated with the present value of the length of the band. That is, the stretch is really a local property of the deformation function describing the configurations of the band, and may be correlated with the end-to-end length provided that it is uniformly distributed.

Because the length of a segment is arbitrary in principle, we may pass to the limit and associate the response with a point of the material, defined as the limit of a sequence of intervals whose lengths tend to zero. In this way, we associate the global force-extension response with properties of the material per se, presumed to be operative on an arbitrarily small length scale. This is one of the premises of continuum theory; namely, that the properties of the material are assigned to points of the continuum. These days, thanks to the rise of computing, it is often augmented by the notion of a hierarchy of continua that operate at length scales smaller than the unaided eye would associate with a point. In some cases, the smaller scale continua are replaced by discrete or finite-dimensional, systems. Some form of communication among the length scales is then required, furnishing material properties on the larger scales in terms of system response on the smaller scales. The basic idea is known collectively as the *multi-scale* method. While it is currently an active field producing interesting and sometimes useful work, it is not a panacea for the limitations of conventional continuum theory, but rather a way of exploiting computing power to avoid the empirical work that standard continuum theory requires to realize its full potential. Recalling the dubious effectiveness of early formulations of rubber elasticity based on statistical mechanics, it is perhaps not surprising that multiscale methods are typically no more reliable than basic continuum theory, while often requiring the use of ever more models operating at ever finer length scales. All is well and good if this process converges in some sense, but whether or not it does depends as much on the problem being addressed as anything else, and in any case the issue is almost never explored carefully. This brings to mind Truesdell's (1984) amusing observation to the effect that continuum theory is immunized by its very nature against the next great discovery in atomic physics, remaining indifferent to the parade of sub- and sub-sub-atomic particles that blink in and out of existence while we labor over our engineering calculations, oblivious to their comings and goings. We digress, however.

If the material is non-uniform, little can be concluded from the simple rubber-band experiment about the nature of the *material*, in contrast to the situation for uniform materials that are uniformly stretched. This is due to the fact that the stretch will now be non-uniform, despite the uniformity of the force intensity (by virtue of equilibrium), due to the variations in the way the material responds locally to that force. In this case, we perform a sequence of experiments on ever shorter segments. If the force-deflection curves thus obtained converge, then the limit response may be said to characterize the material at the length scale associated with the last segment in the sequence. This generates the response $f = F(\lambda; x)$, where x is the location from one end, say, of the band to a point contained in the intersection of the sequence of the segments prior to deformation. Here the stretch is now a function of x, and the force required to maintain it at the value $\lambda(x)$ will reflect the non-uniformity of the material; hence, the explicit dependence of F on x. Because the value of this function—the force—is uniform in the equilibrated band, it is the *stretch* that must adjust to the non-uniformity of the material, producing a field $\lambda(x)$. If the sequence of tests does not converge, then we assign the response $F(\lambda; x)$ to the one *point* x that remains as the segment lengths diminish to zero. In principle, if not in practice, this is the sort of thing one does to test for material non-uniformity and to quantify the associated response function.

The attendant difficulties may indeed give some impetus to multiscale methods, but it bears repeating that these entail the use of models in lieu of actual data. Such practices are fraught with their own difficulties, not least among them being the need for empirical testing of the models purportedly operating on the smaller scales. In any case, as for uniform bands, we have a relation for the force that depends on the present values of the stretch function, but not on the history of the deformation, or on how quickly or slowly it occurs. This is due to our restriction to time scales on which the force and stretch are sensibly constant, and is what most people mean by the word *elasticity*.

Interestingly, a relation of much the same kind is found when the band is deformed very rapidly, as when a wave is caused to propagate through it. This is due to the fact that the deformation then occurs on a time scale that is too short for effects like viscosity to be effective in relaxing the force, whereas in the slow experiment in which data are recorded on long time scales, such effects have already run their course. Behavior on intermediate time scales, which is the province of viscoelasticity theory, is beyond the scope of this book. An up-to-date account, accessible to students of nonlinear elasticity, may be found in the article by Wineman (2009).

A principal lesson of the rubber-band experiment is that material response appears to be local. That is, the force acting on a short segment of the band, and hence (by equilibrium) the force at any cross section within it, is determined by the deformation of that segment and not by the deformations of other segments comprising the original band. This idea has been codified by Noll in his *principle of local action.* Current pedagogy tends to discourage the use of this kind of language, as it seems to confer special status on simple ideas. Nevertheless, this principle furnishes a logical point of departure for the abstraction of simple experiments of the foregoing kind. It exemplifies the kind of fundamental reasoning that is needed to move from observations to a conception of how Nature works, and from there to the formulation of a predictive theory, which is surely among the noblest aspirations of Man. In our case, this takes the form of an assumption to the effect that the Cauchy stress at a material point, labelled p, say, is sensitive only to the deformation of material points in its vicinity (see Figure 1.2).

More precisely, the Cauchy stress $\mathbf{T}(p, t)$ is determined by the deformation $\chi(\mathbf{x}', t)$ for those reference positions $\mathbf{x}'$ satisfying $\mathbf{x}' \in N_\kappa(p)$, a neighborhood of the material point p of the body occupying position $\mathbf{x}$ in reference configuration κ. Here $\mathbf{y} = \chi(\mathbf{x}, t)$ is the position in three-space at time t associated with the same material point. In the older literature one often sees the word "particles" used in place of our "points." However, the former connotes a collection of discrete objects, which is not what is intended when using continuum theory.

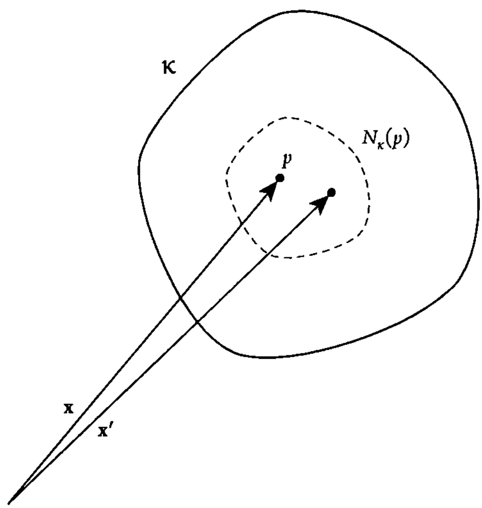

Figure 1.2 Neighborhood of a material point, p, in the reference configuration

It is appropriate to append a subscript to the deformation function and write $\mathbf{y} = \chi_\kappa(\mathbf{x}, t)$, to acknowledge the choice of reference. The fact that current position is unrelated to the reference implies that the *function* taking $\mathbf{x}$ to $\mathbf{y}$ necessarily depends on this choice. This issue will be revisited later. The Cauchy stress is measurable without reference to κ, both in principle and in practice, and so a subscript would not be appropriate.

In summary, we suppose that $\mathbf{T}(p, t)$ is determined by $\chi_\kappa(\mathbf{x}', t)$ for $\mathbf{x}' \in N_\kappa(p)$, which contains the position $\mathbf{x}$ of p in κ.

If the deformation function is smooth in its first argument, and if the diameter of $N_\kappa(p)$ is suitably small, then the deformations that determine $\mathbf{T}(p, t)$ may be approximated by

$$\chi_\kappa(\mathbf{x}', t) = \chi_\kappa(\mathbf{x}, t) + \mathbf{F}_\kappa(\mathbf{x}, t)(\mathbf{x}' - \mathbf{x}) + o(|\mathbf{x}' - \mathbf{x}|), \tag{1.1}$$

where $\mathbf{F}_\kappa$, called the *deformation gradient*, is the derivative of the function $\chi_\kappa(\mathbf{x}, t)$ with respect to its first argument. This should carry the subscript in principle, but in practice it is cumbersome to do so and for the most part we shall not. The small "oh" identifies terms that tend to zero faster than the argument does, as the latter tends to zero. For points sufficiently close to the place $\mathbf{x}$ occupied by p in the reference, the response is then determined primarily by $\chi_\kappa(\mathbf{x}, t)$ and $\mathbf{F}_\kappa(\mathbf{x}, t)$. If one is concerned with leading-order effects, then it would be sensible to retain only the first term in (eqn 1.1) and consider a model in which the stress at p is sensitive only to $\chi_\kappa(\mathbf{x}, t)$. However, we will see that such dependence is precluded by invariance arguments and so the actual leading term is the deformation gradient. Retention of this term alone leads to a famous model for materials named the *simple material* by Noll, who advanced the idea not only for elasticity, but for other theories in continuum mechanics as well.

One can, of course, envisage applications in which retention of further terms is appropriate, the next one being the gradient of $\mathbf{F}_\kappa(\mathbf{x}, t)$. The model thus derived turns out to be rather useful for describing localized effects such as surface tension in solids. More recently, it has been used to model materials reinforced by a dense distribution of fibers in which the fibers are presumed to offer elastic resistance to flexure. Flexure is nothing more than the curvature induced by deformation, while curvature is determined by the second derivative of the position function on a fiber with respect to arc length; this in turn is determined by both the deformation gradient and its gradient. Having simpler applications in mind, we do not study this relatively complex model here. The interested reader will find excellent treatments of it in papers by Toupin (1962, 1964) and by Spencer and Soldatos (2007).

The model we intend to study is thus of the form

$$\mathbf{T}(p, t) = \mathcal{G}_\kappa(\chi_\kappa(\mathbf{x}, t), \mathbf{F}_\kappa(\mathbf{x}, t); \mathbf{x}), \tag{1.2}$$

in which the last argument is intended to indicate a parametric dependence on the material point and, hence, on its position in κ. This is needed if the properties of the material, codified in the *constitutive* function $\mathcal{G}_\kappa(\cdot, \cdot; \mathbf{x})$, vary from point to point; that is, if the material is non-uniform. In principle, this function is determined by experiments, but these are cumbersome and expensive, and so before going to the laboratory we should try to simplify it as far as possible. The manner of doing just that comprises much of the theoretical underpinnings of the subject. We might have included a dependence on time, but as we shall see presently this too is disallowed by invariance arguments. We note, however, that the subscript κ on the constitutive function is required, because this function depends on variables associated with a reference placement whereas its values do not; the *function* itself must then depend on the reference. The reference is not prescribed *for* us, but instead is specified *by* us, subject to the requirement that positions within it be in one-to-one correspondence with points of the material. Almost always workers in the subject confine attention to references that are, or could be, occupied by the material in the course of its motion, this carrying the mild restriction

$$J > 0, \quad \text{where} \quad J = \det \mathbf{F}_\kappa. \tag{1.3}$$

Changing the reference means changing the constitutive function in such a way as to leave the Cauchy stress invariant. After all, experiments designed to measure the Cauchy stress do not require knowledge of our idiosyncratic choice of reference. In this way, given the constitutive function based on a particular choice of reference, we can compute that which applies to any other admissible choice. The reader is cautioned that long-standing practice is to associate the reference with a stress-free configuration of the material. Not only does this promote the erroneous view that the reference needs to have some special physical status, it also demands that we accept the fiction that the existence of global stress-free states is the norm, rather than the exception. We will take up this issue in Chapter 13.

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2

Observers and invariance

The historical development of Physics has been guided by one overarching idea: that the laws of Nature have nothing to do with us and, hence, that the mathematical description of these laws should satisfy invariance requirements representing such indifference in mathematical terms. This egalitarian, as distinct from egocentric philosophy marks the development of Physics just as surely as it characterizes the healthy psychological development of human beings. Alas, as with all that seems obvious, it must face certain challenges; in this instance that offered by the *uncertainty principle* of quantum mechanics, which teaches that the act of making an observation has a non-negligible effect on that which is observed. (See the book by Murdoch (2012) for an interesting discussion.) We shall not, however, develop elasticity theory from the quantum mechanical point of view here, despite promising developments emerging from current research.

To understand the consequences of the idea of material indifference for elasticity theory, it is necessary to admit different points of view, or *observers*, into contention so that we may know what it is about them that should not influence a sound physical theory. For example, in Relativity Theory an observer is identified with a frame of reference. Observers have little in common except for their agreement on one thing—the speed of light in vacuum. Accordingly, the speed of light in vacuum is said to be *frame invariant*. This seemingly innocuous constraint on the laws of physics has the most profound mathematical consequences, known collectively as the Theory of Relativity. Classical Mechanics, to which attention is confined here and in most treatments of continuum mechanics, is based on a similar idea, except that classical observers are presumed to agree on two things—the distance between any pair of material points and the time lapse between events. A penetrating discussion may be found in a paper by Noll (1973).

This is not all, however. Following an important paper by Murdoch (2003), we suppose that observers also agree on the nature of the material. In our case, that it is elastic, and hence on the list of variables (e.g., the Cauchy stress, the deformation, and the deformation gradient in the case of elasticity) that are related by the constitutive equations pertaining to any observer. After all, the manner in which a sample of material responds to stimuli is presumably unaffected by the observer of such response; and so, if a particular set of variables is found by one observer conducting an experiment to be relevant then it should be so for all. We are belaboring this matter perhaps more than we should, because as reasonable

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as the concept may appear to the uninitiated, it has been the cause of considerable confusion and suffering among the educated. My own not inconsiderable reading leads me to the view that the interpretation offered by Noll and Murdoch is superior to the alternatives as far as classical mechanics is concerned. To this day, workers are often divided over this issue along the party lines that have emerged during the modern development of our subject.

The relationship between a pair of classical observers, $\mathcal{O}$ and $\mathcal{O}^+$, say, may be expressed in the form (see Figure 2.1)

$$\chi^+_{\kappa^+}(\mathbf{x}^+, t^+) = \mathbf{Q}(t)\chi_\kappa(\mathbf{x}, t) + \mathbf{c}(t) \quad \text{and} \quad t^+ = t + a, \tag{2.1}$$

wherein t is the time on the watch, κ is the reference configuration used, and $\chi_\kappa(\mathbf{x}, t)$ is the deformation, all pertaining to $\mathcal{O}$; whereas the same variables, carrying the superscript $^+$, pertain to $\mathcal{O}^+$. Here $\mathbf{Q}(t)$ is a time-dependent orthogonal tensor ($\mathbf{Q}^t\mathbf{Q} = \mathbf{Q}\mathbf{Q}^t = \mathbf{I}$, the identity tensor), $\mathbf{c}(t)$ is a vector-valued function, and a is a constant. The ideas underpinning this relation are explained in Noll's paper. We note that it is quite similar to the relation existing between a deformation as perceived by one observer and a second deformation, perceived by the *same* observer, obtained by superposing a rigid-body deformation on the first. However, in the latter the orthogonal transformation is required to be proper-orthogonal, whereas in eqn (2.1) it is not. We shall return to this point presently. Basically, eqn (2.1), part 1, ensures that the distance between material points p_1 and p_2 at a particular instant is the same for both observers, whereas eqn (2.1), part 2, ensures that the time lapse between successive events is likewise the same. Indeed, eqn (2.1) is necessary and sufficient for such agreement. We note that the two observers are free not only to wear different watches, but also to choose different references. Before Murdoch, the literature was marred by the frequent repetition of the unnatural view that these references may be assumed to coincide.

In concert with eqn (2.1), we suppose, this time truly without loss of generality (see the Problems), that the configuration $R_{|t_0}$, say, occupied by the body at time t_0, is chosen by

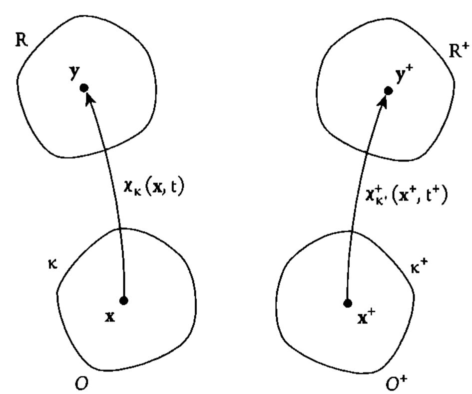

Figure 2.1 Configuration of a body, as perceived by observers O and O^+

observer $\mathcal{O}$ to serve as reference; in short, $\kappa = R_{|t_0}$. We further suppose that observer $\mathcal{O}^+$ takes up the same suggestion and selects $\kappa^+ = R^+_{|t_0^+}$. Then, $\det \mathbf{F}^+_{\kappa^+} > 0$, while (2.1) requires

$$\mathbf{x}^+ = \mathbf{K}\mathbf{x} + \mathbf{c}_0, \tag{2.2}$$

wherein $\mathbf{K} = \mathbf{Q}(t_0)$, etc. The Chain Rule (see Supplemental Notes, Part 3) yields the chain of equalities:

$$d\mathbf{y}^+ = \mathbf{Q}d\mathbf{y} = \mathbf{Q}\mathbf{F}_\kappa d\mathbf{x} = \mathbf{Q}\mathbf{F}_\kappa \mathbf{K}^t d\mathbf{x}^+, \quad \text{and thus} \quad \mathbf{F}^+_{\kappa^+} = \mathbf{Q}\mathbf{F}_\kappa \mathbf{K}^t, \tag{2.3}$$

the latter implying that

$$(\det \mathbf{Q})(\det \mathbf{K}) = 1. \tag{2.4}$$

In view of our proposal regarding the two observers' perceptions of material response, the relevant constitutive relation for $\mathcal{O}^+$ is necessarily of the form (cf. eqn (1.2))

$$\mathbf{T}^+(p, t^+) = \mathcal{G}^+_{\kappa^+}(\chi^+_{\kappa^+}(\mathbf{x}^+, t^+), \mathbf{F}^+_{\kappa^+}(\mathbf{x}^+, t^+); \mathbf{x}^+). \tag{2.5}$$

In every course on continuum mechanics, and in Murdoch's paper, it is explained that $\mathbf{T}^+(p, t^+) = \mathbf{Q}(t)\mathbf{T}(p, t)\mathbf{Q}(t)^t$; therefore,

$$\mathcal{G}^+_{\kappa^+}(\chi^+_{\kappa^+}(\mathbf{x}^+, t^+), \mathbf{F}^+_{\kappa^+}(\mathbf{x}^+, t^+); \mathbf{x}^+) = \mathbf{Q}(t)\mathcal{G}_\kappa(\chi_\kappa(\mathbf{x}, t), \mathbf{F}_\kappa(\mathbf{x}, t); \mathbf{x})\mathbf{Q}(t)^t, \tag{2.6}$$

and this holds for all orthogonal $\mathbf{Q}(t)$, for all $\mathbf{c}(t)$ and for all a.

To obtain necessary conditions, consider a situation in which $\chi^+_{\kappa^+}$ and $\mathbf{F}^+_{\kappa^+}$ are observed by $\mathcal{O}^+$ to persist at fixed values during the interval $[t_1^+, t_2^+]$. This observer perceives a static configuration of the body, while the other observer is flying past in an airplane, say, all the while observing the same body. Evaluating (2.6) at times $t_1 = t_1^+ - a$ and $t_2 = t_2^+ - a$ and eliminating $\mathcal{G}^+_{\kappa^+}(\chi^+_{\kappa^+}, \mathbf{F}^+_{\kappa^+}; \mathbf{x}^+)$, we derive

$$\mathbf{Q}_1\mathcal{G}_\kappa(\mathbf{y}_1, \mathbf{F}_1; \mathbf{x})\mathbf{Q}^t_1 = \mathbf{Q}_2\mathcal{G}_\kappa(\mathbf{y}_2, \mathbf{F}_2; \mathbf{x})\mathbf{Q}^t_2, \tag{2.7}$$

where $\mathbf{y}_1 = \chi_\kappa(\mathbf{x}, t_1)$, $\mathbf{F}_1 = \mathbf{F}_\kappa(\mathbf{x}, t_1)$, $\mathbf{Q}_1 = \mathbf{Q}(t_1)$, etc. This is a restriction on the constitutive function used by $\mathcal{O}$. Furthermore,

$$\mathbf{Q}_1\mathbf{y}_1 + \mathbf{c}_1 = \mathbf{Q}_2\mathbf{y}_2 + \mathbf{c}_2, \tag{2.8}$$

where $\mathbf{c}_1 = \mathbf{c}(t_1)$, etc., and from $(2.3)_2$ we have $\mathbf{Q}_1\mathbf{F}_1\mathbf{K}^t = \mathbf{Q}_2\mathbf{F}_2\mathbf{K}^t$, which furnishes

$$\mathbf{F}_2 = \mathbf{Q}\mathbf{F}_1, \quad \text{where} \quad \mathbf{Q} = \mathbf{Q}^t_2\mathbf{Q}_1. \tag{2.9}$$

Here, of course, $\mathbf{F}_1$ and $\mathbf{F}_2$ are the values of $\mathbf{F}_\kappa$ at different times and so (1.3) requires that:

$$\mathbf{Q} \in Orth^+, \tag{2.10}$$

the set of proper-orthogonal (rotation) tensors; i.e., the orthogonal tensors with determinant +1. Consequently eqn (2.7) may be re-written as

$$\mathcal{G}_\kappa(\mathbf{Q}\mathbf{y} + \mathbf{d}, \mathbf{Q}\mathbf{F}; \mathbf{x}) = \mathbf{Q}\mathcal{G}_\kappa(\mathbf{y}, \mathbf{F}; \mathbf{x})\mathbf{Q}^t, \quad \text{where} \quad \mathbf{d} = \mathbf{Q}_2^t\mathbf{c} \quad \text{with} \quad \mathbf{c} = \mathbf{c}_1 - \mathbf{c}_2. \tag{2.11}$$

This is precisely the requirement that the constitutive function $\mathcal{G}_\kappa$ pertaining to $\mathcal{O}$ be invariant under superposed rigid-body motions.

The reader is cautioned that such invariance, rather than observer consensus regarding material response, is used by many workers as a basic premise regarding constitutive behavior. Murdoch was the first to show that the latter implies the former, and the demonstration of this implication given here closely follows his work. Indeed, it is difficult to understand why one would impose invariance under superposed rigid motions as a primitive hypothesis. For, the response of materials is constrained by the equations of motion

$$div\mathbf{T} + \rho\mathbf{b} = \rho\ddot{\mathbf{y}} \quad \text{and} \quad \mathbf{T} = \mathbf{T}^t, \tag{2.12}$$

where ρ is the spatial mass density, $\mathbf{b}$ is the body force per unit mass, superposed dots refer to material time derivatives ($\partial/\partial t$ at fixed $\mathbf{x}$) and div is the spatial divergence; that is, the divergence based on position $\mathbf{y}$. These imply that an inertial force is imposed on the material when the body is subjected to a rigid-body motion superposed on any given motion; indeed, one would generally need to supply a rather strange distribution of body force to maintain rigidity of the superposed motion. While such might be produced at a given point of the body, it is extremely unlikely that it could be generated globally. Even if it could, there seems to be no reason to suppose *a priori* that the material responds to such forces in accordance with (2.11). In contrast, the alternative view, based on observer consensus, imposes no restrictions on material behavior apart from agreement on the kind of response (here, elastic) that is elicited. Beyond this conceptual advantage, this point of view is in harmony with ideas underlying Relativistic Physics, which in principle should subsume Classical Mechanics.

As a further caution we point out that occasional critics of Murdoch's reasoning object that eqn (2.6) yields the conclusion that the constitutive equation for $\mathcal{O}^+$, say, necessarily changes as his/her motion evolves relative to $\mathcal{O}$. In other words, if $\mathcal{O}$ is entitled to the use of some fixed constitutive function, then $\mathcal{O}^+$ is not and must therefore be expected to keep close track of $\mathcal{O}$. On the contrary, eqn (2.6) merely imposes a restriction on the constitutive equations used by the two observers so as to ensure their agreement, if indeed they are ever consulted, about the nature of material response. We return to this point below.

Continuing, we have arrived at eqn (2.11) as a logical consequence of eqn (2.6). To explore the potential for further consequences, consider the special case $\mathbf{Q}_2 = \mathbf{Q}_1 = \pm\mathbf{I}$. Then, $\mathbf{Q} = \mathbf{I}$ and (2.11) reduces to

$$\mathcal{G}_\kappa(\mathbf{y} + \mathbf{d}, \mathbf{F}; \mathbf{x}) = \mathcal{G}_\kappa(\mathbf{y}, \mathbf{F}; \mathbf{x}), \tag{2.13}$$

implying that the constitutive function is unaffected by arbitrary variations in its first argument; i.e., that it is *translation invariant*. It is thus independent of that argument, and so we arrive at the major simplification

$$\mathcal{G}_\kappa(\mathbf{y}, \mathbf{F}; \mathbf{x}) = \mathbf{G}_\kappa(\mathbf{F}; \mathbf{x}), \tag{2.14}$$

for some function $\mathbf{G}_\kappa$, while (2.11) reduces to

$$\mathbf{G}_\kappa(\mathbf{Q}\mathbf{F}; \mathbf{x}) = \mathbf{Q}\mathbf{G}_\kappa(\mathbf{F}; \mathbf{x})\mathbf{Q}^t \quad \text{for all} \quad \mathbf{Q} \in Orth^+. \tag{2.15}$$

If we use the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$, where $\mathbf{R}$ is a rotation and $\mathbf{U}$ is the unique positive-definite symmetric tensor satisfying $\mathbf{U}^2 = \mathbf{F}^t\mathbf{F}$, and if we choose $\mathbf{Q} = \mathbf{R}^t_{|p}$, it follows that

$$\mathbf{G}_\kappa(\mathbf{F}; \mathbf{x}) = \mathbf{R}\mathbf{G}_\kappa(\mathbf{R}^t\mathbf{F}; \mathbf{x})\mathbf{R}^t = \mathbf{R}\mathbf{G}_\kappa(\mathbf{U}; \mathbf{x})\mathbf{R}^t = \mathbf{F}\hat{\mathbf{G}}_\kappa(\mathbf{U}; \mathbf{x})\mathbf{F}^t,$$
$$\text{where} \quad \hat{\mathbf{G}}_\kappa(\mathbf{U}; \mathbf{x}) = \mathbf{U}^{-1}\mathbf{G}_\kappa(\mathbf{U}; \mathbf{x})\mathbf{U}^{-1}. \tag{2.16}$$

This choice of $\mathbf{Q}$ yields a rotation that depends on t alone, and is therefore admissible in (2.15). In practice, it is usually inconvenient to compute $\mathbf{U}$ from $\mathbf{F}$, whereas it is trivial to evaluate the Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^t\mathbf{F}$. Using the fact that $\mathbf{U}$ is uniquely determined by $\mathbf{C}$, we then have:

$$\mathbf{G}_\kappa(\mathbf{F}; \mathbf{x}) = \mathbf{F}\mathbf{H}_\kappa(\mathbf{C}; \mathbf{x})\mathbf{F}^t, \tag{2.17}$$

where $\mathbf{H}_\kappa(\mathbf{C}; \mathbf{x}) = \hat{\mathbf{G}}_\kappa(\sqrt{\mathbf{C}}; \mathbf{x})$. Thus, we have reached the remarkable conclusion that observers agree on the nature of material response only if the constitutive equation pertaining to any one of them is determined entirely by the symmetric right Cauchy–Green deformation tensor.

We have shown, by special choice of $\mathbf{Q}$, that (2.17) follows from (2.15); i.e., that (2.17) is necessary for (2.15). To show that it is sufficient, we use it to obtain

$$\mathbf{G}_\kappa(\mathbf{Q}\mathbf{F}; \mathbf{x}) = \mathbf{Q}\mathbf{F}\mathbf{H}_\kappa[(\mathbf{Q}\mathbf{F})^t\mathbf{Q}\mathbf{F}; \mathbf{x}](\mathbf{Q}\mathbf{F})^t = \mathbf{Q}\mathbf{F}\mathbf{H}_\kappa(\mathbf{F}^t\mathbf{F}; \mathbf{x})\mathbf{F}^t\mathbf{Q}^t = \mathbf{Q}\mathbf{G}_\kappa(\mathbf{F}; \mathbf{x})\mathbf{Q}^t, \tag{2.18}$$

which is valid for any rotation $\mathbf{Q}$. Therefore, eqns (2.15) and (2.17) are equivalent. Hence, we conclude that the Cauchy stress at material point p is given by a function of the form

$$\mathbf{T}(p, t) = \mathbf{F}\mathbf{H}_\kappa(\mathbf{C}; \mathbf{x})\mathbf{F}^t. \tag{2.19}$$

Euler's laws of motion require that $\mathbf{T}$ be symmetric, and $\mathbf{H}_\kappa$ therefore takes values in the set of symmetric tensors; the Cauchy stress is completely determined by a symmetric tensor-valued function of a symmetric tensor. From the experimental point of view, this affords a major simplification over the original hypothesis embodied in (1.2). Indeed, reasoning of this kind is one of the hallmarks of modern continuum mechanics and should always be applied before attempting any laboratory assessment of material behavior.

If desired, the constitutive equation used by $\mathcal{O}^+$ may be determined in terms of that used by $\mathcal{O}$. Combining (2.6), (2.14) and (2.17) furnishes

$$\mathcal{G}^+_{\kappa^+} = \mathbf{F}^+\mathbf{H}^+_{\kappa^+}(\mathbf{C}^+; \mathbf{x}^+)(\mathbf{F}^+)^t, \tag{2.20}$$

where

$$\mathbf{H}^{+}_{\kappa^{+}}(\mathbf{C}^{+};\mathbf{x}^{+}) = \mathbf{K}\mathbf{H}_{\kappa}(\mathbf{K}^{t}\mathbf{C}^{+}\mathbf{K};\ \mathbf{K}^{t}(\mathbf{x}^{+} - \mathbf{c}_0))\mathbf{K}^{t} \tag{2.21}$$

in which $\mathbf{K}$ and $\mathbf{c}_0$ are fixed parameters. This constitutive *function* is fixed once and for all, and depends on the same list of variables, as interpreted by $\mathcal{O}^{+}$, as those involved in the relation used by $\mathcal{O}$.

Other stress measures are of use in the formulation of elasticity theory. They may be defined in terms of their connections to the Cauchy stress. For example, the popular Piola stress, $\mathbf{P}$, is given by

$$\mathbf{P} = \mathbf{T}\mathbf{F}^{*}, \tag{2.22}$$

where $\mathbf{F}^{*}$ is the cofactor of $\mathbf{F}$ defined by

$$\mathbf{F}^{*}(\mathbf{a} \times \mathbf{b}) = \mathbf{F}\mathbf{a} \times \mathbf{F}\mathbf{b} \tag{2.23}$$

for all three-vectors $\mathbf{a}$ and $\mathbf{b}$; this may be used with eqn (1.3) to show that

$$\mathbf{F}^{*} = J\mathbf{F}^{-t}, \tag{2.24}$$

provided that $\mathbf{F}$ is invertible, as we have assumed. Whether or not this is the case, it is possible to show that the Cartesian components satisfy

$$F^{*}_{iA} = \frac{1}{2}e_{ijk}e_{ABC}F_{jB}F_{kC}, \tag{2.25}$$

where e is the permutation symbol ($e_{123} = +1$, etc.). See Part 1 of the Supplemental Notes.

In addition, the second Piola–Kirchhoff stress, $\mathbf{S}$, is defined by

$$\mathbf{P} = \mathbf{F}\mathbf{S}. \tag{2.26}$$

These stresses should carry the subscript κ in principle, as is clear from their definitions, but to avoid cluttered notation we shall invoke our policy regarding $\mathbf{F}$ and, thus, usually refrain from doing so. Using the definitions, it is easy to show that the symmetry of $\mathbf{T}$ is equivalent to that of $\mathbf{S}$. Using eqns (2.19), (2.22), and (2.26), we also have

$$\mathbf{S} = \hat{\mathbf{S}}_{\kappa}(\mathbf{C};\mathbf{x}), \quad \text{where} \quad \hat{\mathbf{S}}_{\kappa}(\mathbf{C};\mathbf{x}) = \sqrt{\det\mathbf{C}}\,\mathbf{H}_{\kappa}(\mathbf{C};\mathbf{x}), \tag{2.27}$$

and

$$\mathbf{P} = \hat{\mathbf{P}}_{\kappa}(\mathbf{F};\mathbf{x}), \quad \text{where} \quad \hat{\mathbf{P}}_{\kappa}(\mathbf{F};\mathbf{x}) = \mathbf{F}\hat{\mathbf{S}}_{\kappa}(\mathbf{F}^{t}\mathbf{F};\mathbf{x}). \tag{2.28}$$

The Piola stress is useful because the equation of motion may be expressed concisely in terms of it as

$$Div\mathbf{P} + \rho_\kappa \mathbf{b} = \rho_\kappa \ddot{\mathbf{y}}, \tag{2.29}$$

where Div is now the referential divergence (based on $\mathbf{x}$) and $\rho_\kappa = J\rho$ is the referential mass density. Conservation of mass - the notion that the mass of a set of material points remains always invariant - is expressible concisely as $\dot{\rho}_\kappa = 0$.

In the old days some workers were seemingly put off by the fact that the Piola stress, by virtue of eqns (2.22) and (2.26), is not symmetric. They tended to prefer the second Piola–Kirchhoff stress for this reason. Of course there is no free lunch and the equation of motion based on the second Piola–Kirchhoff stress, given by substituting eqns (2.26) into (2.29), is seen, unlike eqns (2.12) or (2.29), to involve the deformation explicitly. This is of no consequence whatsoever, either to the theory or to its implementation, and we shall not belabor it further.

Problems

1. Given the (Cauchy) stress-response function $\mathbf{G}_1(\mathbf{F}_1; \mathbf{x}_1)$, and a differentiable map $\mathbf{x}_2 = \lambda(\mathbf{x}_1)$ from reference configuration κ_1 to reference configuration κ_2, derive the constitutive function $\mathbf{G}_2(\mathbf{F}_2; \mathbf{x}_2)$.
2. Repeat the argument about observer consensus, this time without requiring the observers to choose some *initial* configurations as reference, to derive the appropriate restriction on $\mathcal{G}_\kappa(\chi, \mathbf{F}; \mathbf{x})$. Clearly point out any changes in the argument, and whether or not the final result is different from eqn (2.19). Note that the references are only required to be in one-to-one correspondence with those adopted in the text.
3. How does the argument change if an observer decides to switch to the use of a mirror to observe the body at some instant in a specified time interval? Of course, this happens every day in many branches of science.
4. Write the balance law (2.12) in global form and use Nanson's formula:

$$\alpha \mathbf{n} = \mathbf{F}^* \mathbf{N}, \tag{2.30}$$

where $\mathbf{N}$ and $\mathbf{n}$ respectively are the unit normals to a material surface in the reference and current configurations, and α is the ratio of the area measures of the surface in the current placement to that in the reference, to derive a global form of the equation involving integration over the reference. Localize this to obtain eqn (2.29). (A proof of the so-called *localization theorem*, which is one of the main tools of continuum mechanics, may be found in the book by Gurtin, 1981.).

5. Prove the Piola identity $Div\mathbf{F}^* = \mathbf{0}$. Hint: use the result of the previous problem together with the divergence and localization theorems. Alternatively, with reference to eqn (2.25), use the fact that $F^*_{iA} = \psi_{iAB,B}$, where $\psi_{iAB} = \frac{1}{2} e_{ijk} e_{ABC} \chi_j \chi_{k,C} = -\psi_{iBA}$.

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3

Mechanical power and hyperelasticity

3.1 Elasticity and energy

The well-known mechanical power identity of conventional continuum mechanics follows on scalar-multiplying (2.12) by the velocity $\dot{\mathbf{y}}$, integrating over an arbitrary set S of material points, and applying the divergence theorem. Thus,

$$\mathcal{P}(S,t) = \mathcal{S}(S,t) + \frac{d}{dt}\mathcal{K}(S,t), \tag{3.1}$$

where

$$\mathcal{P}(S,t) = \int_{\partial P} \mathbf{t} \cdot \dot{\mathbf{y}} da + \int_P \rho \mathbf{b} \cdot \dot{\mathbf{y}} dv, \quad \mathcal{S}(S,t) = \int_P \mathbf{T} \cdot \mathbf{L} dv \tag{3.2}$$

and

$$\mathcal{K}(S,t) = \frac{1}{2}\int_P \rho \dot{\mathbf{y}} \cdot \dot{\mathbf{y}} dv \tag{3.3}$$

are respectively the power supplied to, the stress power in, and the kinetic energy of S, which occupies the volume P at time t. Here, P is a subset of the region of space R occupied by the entire body at time t, $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ is the spatial velocity gradient, and

$$\mathbf{t} = \mathbf{Tn}, \tag{3.4}$$

where $\mathbf{n}$ is the exterior unit-normal field on the boundary ∂P, is the traction, or contact force per unit area, transmitted to S by the environment. The dot between vectors is, of course, the usual Euclidean inner product, while that between tensors is defined by $\mathbf{A} \cdot \mathbf{B} = tr(\mathbf{AB}^t)$, where $tr(\cdot)$ is the usual trace operation. This extends the definition of the Euclidean inner product to (2nd order) tensors; in fact, it is seen to be identical to the usual vector definition when written out in terms of components on an orthonormal basis.

Finite Elasticity Theory. David J. Steigmann.

The balance (3.1) presumes conservation of mass; that is,

$$\frac{d}{dt}M(S) = 0, \quad \text{where} \quad M(S) = \int_P \rho dv = \int_\pi \rho_\kappa dv \tag{3.5}$$

and π is the subset of the fixed region κ associated with S.

The balance given in eqn (3.1) differs in form from that associated with a discrete particle. The latter is $\mathcal{P} = d\mathcal{K}/dt$, where $\mathcal{P}$ and $\mathcal{K}$ are the power supplied to the particle, by the net force acting on it, and its kinetic energy, respectively; there being, of course, no analog of stress power. For example, if the particle is a mass tethered to an elastic spring and acted upon by an applied force, undergoing a one-dimensional motion $y(t)$ while maintaining frictionless contact with a horizontal plane, then the energy balance takes the special form $\mathcal{P} = d\mathcal{E}/dt$, where $\mathcal{E} = \mathcal{K} + \mathcal{U}$ in which $\mathcal{U}$ is the spring energy, obtained by integrating the spring force $F(y) = -\mathcal{U}'(y)$, leaving unspecified an irrelevant constant of integration. Given $F(y)$, such integration is always possible and yields, in the case of unforced motion, the conservation law $d\mathcal{E}/dt = 0$, expressing the fact that the total mechanical energy remains fixed in the course of the motion.

Proceeding by analogy we suppose that elastic bodies are like elastic springs and that a similar conservation law holds for them in the case of unforced motion. Thus we assume the existence of an energy $\mathcal{U}$ such that the stress power is expressible as $\mathcal{S} = d\mathcal{U}/dt$, yielding the conservation law $d\mathcal{E}/dt = 0$ with $\mathcal{E} = \mathcal{K} + \mathcal{U}$; this time, of course, for the continuum instead of the particle. Forced motions are then such as to satisfy $\mathcal{P} = d\mathcal{E}/dt$. We assume $\mathcal{U}$ to be an absolutely continuous function; here, of volume, and thus suppose that

$$\mathcal{U}(S,t) = \int_\pi W dv, \tag{3.6}$$

where W is the (referential) strain-energy density.

We know, from eqn (3.2), that the stress power is expressible in terms of the stress and the rate of deformation. Using the connection eqn (2.22) between the Cauchy and Piola stresses, and the formula given in eqn (2.24), we derive

$$\mathcal{S}(S,t) = \int_\pi J\mathbf{T} \cdot \dot{\mathbf{F}}\mathbf{F}^{-1} dv = \int_\pi tr(J\mathbf{T}\mathbf{F}^{-t}\dot{\mathbf{F}}^t) dv = \int_\pi \mathbf{P} \cdot \dot{\mathbf{F}} dv, \tag{3.7}$$

and, therefore, according to the analogy,

$$\int_\pi \mathbf{P} \cdot \dot{\mathbf{F}} dv = \int_\pi \dot{W} dv. \tag{3.8}$$

Because $\pi \subset \kappa$ is arbitrary, we may localize and use eqn (2.28) to conclude that

$$\hat{\mathbf{P}}_\kappa(\mathbf{F}; \mathbf{x}) \cdot \dot{\mathbf{F}} = \dot{W}, \tag{3.9}$$

pointwise in κ. For this to make sense it must be possible to integrate along a path $\mathbf{F}(t)$, between specified limits, to obtain the difference of the function W determined by those limits and thus depending on the associated values of $\mathbf{F}$. Fixing the lower limit and allowing the upper to be arbitrary, we thereby construct a function $W(\mathbf{F};\mathbf{x})$, to within a function of $\mathbf{x}$ only, such that

$$[\hat{\mathbf{P}}_\kappa(\mathbf{F};\mathbf{x}) - W_{\mathbf{F}}(\mathbf{F};\mathbf{x})] \cdot \dot{\mathbf{F}} = 0, \tag{3.10}$$

where $W_{\mathbf{F}}$ is the tensor-valued derivative of the scalar W with respect to the tensor $\mathbf{F}$ (see Supplemental Notes, Part 2).

The first factor in the inner product is an element of the set of second-order tensors. This is a linear space, just like the space of conventional vectors. We follow common practice and denote it by *Lin*. The second factor is the limit of a difference quotient involving elements of Lin^+, the subset of *Lin* consisting of tensors with positive determinant (see eqn (1.3)). While this is *not* a linear space, the set of differences between its elements is and in fact is just *Lin*. Choosing an arbitrary path $\mathbf{F}(t)$ in Lin^+ containing the point $\mathbf{F}$, we conclude from eqn (3.10) that the term in brackets is orthogonal to any, hence, every element of *Lin* and, therefore, that it vanishes. To see this explicitly we exploit the arbitrariness of $\dot{\mathbf{F}}$ and, after an appropriate scaling of physical units, select $\dot{\mathbf{F}}$ to be the square bracket itself, concluding that its norm, defined by $|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}}$, vanishes and hence that the bracket vanishes, at last yielding

$$\hat{\mathbf{P}}_\kappa(\mathbf{F};\mathbf{x}) = W_{\mathbf{F}}(\mathbf{F};\mathbf{x}). \tag{3.11}$$

Thus, the stress is determined by a scalar-valued function of the deformation gradient, which, like the constitutive equation for the stress, codifies the properties of the particular material at hand. This, too, is therefore a constitutive function, furnishing that for the stress via eqn (3.11). This model is known as *hyperelasticity*. Its antecedent, given by eqn (1.2), is known as *Cauchy elasticity* or simply *elasticity*. Because we have obtained it as a special case, by restricting the theory such that the stress power is expressible as a time derivative, it would appear that hyperelasticity is special. However, we shall see that any elastic material is necessarily hyperelastic when a further condition is imposed that reflects a widespread view about how real materials behave.

Before embarking on this demonstration we digress to consider restrictions on the strain-energy function W following from eqns (2.15), (2.22) and (2.27), which combine to yield

$$\hat{\mathbf{P}}_\kappa(\mathbf{QF}) = \mathbf{Q}\hat{\mathbf{P}}_\kappa(\mathbf{F}) \tag{3.12}$$

for all rotations $\mathbf{Q}$. Here, because we are concerned only with properties pertaining to a material point, we drop explicit reference to $\mathbf{x}$, which plays only a passive role, to help lighten the notation. This will be done henceforth when discussing local properties.

Following Gurtin (1983), consider a path $\mathbf{Q}(t)$ in the set of rotations such that $\mathbf{Q}(0) = \mathbf{I}$ and let $\mathbf{F}(t) = \mathbf{Q}(t)\tilde{\mathbf{F}}$ be an associated path of deformation gradients in which $\tilde{\mathbf{F}}$ is fixed. We are confining attention to a particular material point; the fact that $\mathbf{F}$ is the gradient of a position field does *not* impose any restriction on its values thereat, and so our choice is not subject to any qualifications beyond $\det\tilde{\mathbf{F}} > 0$. In this case (3.12) yields

$$\hat{\mathbf{P}}_\kappa(\mathbf{F}) = \mathbf{Q}\hat{\mathbf{P}}_\kappa(\tilde{\mathbf{F}}), \tag{3.13}$$

and (3.9) implies that

$$\begin{aligned}\dot{W}(\mathbf{F}) &= \hat{\mathbf{P}}_\kappa(\mathbf{F}) \cdot \dot{\mathbf{F}} = \mathbf{Q}\hat{\mathbf{P}}_\kappa(\tilde{\mathbf{F}}) \cdot \dot{\mathbf{Q}}\tilde{\mathbf{F}} = tr[\mathbf{Q}\hat{\mathbf{P}}_\kappa(\tilde{\mathbf{F}})\tilde{\mathbf{F}}^t\dot{\mathbf{Q}}^t] \\ &= tr[\mathbf{\Omega}^t\hat{\mathbf{P}}_\kappa(\tilde{\mathbf{F}})\tilde{\mathbf{F}}^t] = \hat{\mathbf{P}}_\kappa(\tilde{\mathbf{F}})\tilde{\mathbf{F}}^t \cdot \mathbf{\Omega},\end{aligned} \tag{3.14}$$

where $\mathbf{\Omega}^t = \dot{\mathbf{Q}}^t\mathbf{Q}$ is skew and we have used the rule $tr(\mathbf{AB}) = tr(\mathbf{BA})$. But $\hat{\mathbf{P}}_\kappa(\tilde{\mathbf{F}})\tilde{\mathbf{F}}^t$ is just the value of $J\mathbf{T}$ associated with $\tilde{\mathbf{F}}$. Because this is symmetric, the inner product with $\mathbf{\Omega}$ vanishes and it follows that $\dot{W} = 0$. Integrating from $t = 0$ to $t = \tau$, say, we find that

$$W(\tilde{\mathbf{Q}}\tilde{\mathbf{F}}) = W(\tilde{\mathbf{F}}), \tag{3.15}$$

where $\tilde{\mathbf{Q}} = \mathbf{Q}(\tau)$. Because the path in the set of rotations is arbitrary, so is $\tilde{\mathbf{Q}}$ and we conclude that the strain-energy function is insensitive to arbitrary rotations superposed on the given deformation. This invariance is therefore a necessary condition for the symmetry of the Cauchy stress. We drop the tildes and, following on our earlier success, pick $\mathbf{Q} = \mathbf{R}^t_{|p}$, obtaining

$$W(\mathbf{F}) = W(\mathbf{U}) = \hat{W}(\mathbf{C}), \tag{3.16}$$

where $\hat{W}(\mathbf{C}) = W(\sqrt{\mathbf{C}})$, and this, in turn, yields eqn (3.15) for any rotation; eqns (3.15) and (3.16) are, therefore, equivalent. Thus, eqn (3.16) follows from the symmetry of the Cauchy stress.

Substituting eqns (3.11) and (3.16) into eqn (3.9) we find

$$W_{\mathbf{F}} \cdot \dot{\mathbf{F}} = (\hat{W})^{\cdot} = \hat{W}_{\mathbf{C}} \cdot \dot{\mathbf{C}} = Sym\hat{W}_{\mathbf{C}} \cdot \dot{\mathbf{C}}, \tag{3.17}$$

where the prefix *Sym* identifies the symmetric part and $\mathbf{C}(t) = \mathbf{F}(t)^t\mathbf{F}(t)$. This belongs to the set of positive-definite symmetric tensors, while $\dot{\mathbf{C}}$ belongs to the linear space of symmetric tensors. The inner product thus involves only $Sym\hat{W}_{\mathbf{C}}$ and our notation makes this explicit. Alternatively, we may follow common practice and simply define $\hat{W}_{\mathbf{C}}$ to be symmetric. Using $\dot{\mathbf{C}} = \dot{\mathbf{F}}^t\mathbf{F} + \mathbf{F}^t\dot{\mathbf{F}}$ with the rules $\mathbf{A} \cdot \mathbf{BD} = \mathbf{B}^t\mathbf{A} \cdot \mathbf{D} = \mathbf{DA}^t \cdot \mathbf{B}^t$, which follow easily from the properties of the trace operator, we have

$$Sym\hat{W}_{\mathbf{C}} \cdot \dot{\mathbf{F}}^t\mathbf{F} = Sym\hat{W}_{\mathbf{C}} \cdot \mathbf{F}^t\dot{\mathbf{F}} = \mathbf{F}(Sym\hat{W}_{\mathbf{C}}) \cdot \dot{\mathbf{F}}, \tag{3.18}$$

yielding

$$[W_{\mathbf{F}} - 2\mathbf{F}(Sym\hat{W}_{\mathbf{C}})] \cdot \dot{\mathbf{F}} = 0. \tag{3.19}$$

Reasoning as before we conclude that

$$W_{\mathbf{F}} = 2\mathbf{F}(Sym\hat{W}_{\mathbf{C}}), \tag{3.20}$$

and eqns (3.11) and (2.27), part 2, combine to give

$$\hat{\mathbf{S}}_\kappa(\mathbf{C}) = 2Sym\hat{W}_{\mathbf{C}}. \tag{3.21}$$

This, of course, is symmetric and therefore so, too, is the Cauchy stress. Thus, we have shown that eqn (3.15) implies the symmetry of the Cauchy stress. Taken together with our previous result, it follows that such symmetry is *equivalent* to the invariance of the strain energy under superposed rotations.

3.2 Work inequality

Returning to the basis of hyperelasticity, while most of us may be content with the motivation provided by the analogy with springs, we should not ignore objections raised by the skeptics. For them we recount an idea that has become folklore not only in elasticity theory, but in other branches of continuum mechanics as well. Thus, consider the work done on a collection S of material points during a time interval $[t_1, t_2]$. According to eqns (3.1) and (3.7) this is given by

$$\Psi_{12} = \mathcal{K}(S, t_2) - \mathcal{K}(S, t_1) + \int_{t_1}^{t_2} \left(\int_\pi \mathbf{P} \cdot \dot{\mathbf{F}} dv \right) dt. \tag{3.22}$$

Suppose the process is *cyclic* in the sense that the deformation and velocity fields are the same at the start and end of the time interval; that is,

$$\chi(\mathbf{x}, t_1) = \chi(\mathbf{x}, t_2) \quad \text{and} \quad \dot{\chi}(\mathbf{x}, t_1) = \dot{\chi}(\mathbf{x}, t_2), \quad \text{for all} \quad \mathbf{x} \in \kappa. \tag{3.23}$$

Taking gradients, we infer that

$$\mathbf{F}(\mathbf{x}, t_1) = \mathbf{F}(\mathbf{x}, t_2) \quad \text{and} \quad \dot{\mathbf{F}}(\mathbf{x}, t_1) = \dot{\mathbf{F}}(\mathbf{x}, t_2). \tag{3.24}$$

Considering that all points of the body are involved, cyclic processes are no small feat from the experimental point of view, and so our skeptics may not be assuaged after all. We shall therefore resort to regarding such a process as a thought experiment. In general these should be taken with a rather large pinch of salt.

Continuing, we evidently have $\mathcal{K}(S, t_2) = \mathcal{K}(S, t_1)$ in a cyclic process, leaving

$$\Psi_{12} = \int_\pi \left(\int_{t_1}^{t_2} \mathbf{P} \cdot \dot{\mathbf{F}} dt \right) dv, \tag{3.25}$$

where we have interchanged the order of integration, which may be done with impunity if the process is sufficiently smooth (see, for example, Fleming's (1977) book). Intuitively, we expect that it should be necessary to perform non-negative work on a body to cause it to undergo such a process; that is, $\Psi_{12} \geq 0$. This hypothesis is called *the work inequality*.

Not accepting it means having to explain how it is that work can be *extracted* from a body undergoing a cyclic process. Experience suggests that this is futile, and so the hypothesis is widely regarded by the community as being sacrosanct, even though it is really just a thought experiment. In practice, one must contend with instabilities or oscillations that may intervene when one attempts to create a cyclic process from a sequence of homogeneous deformations, these typically causing the deformation to become non-uniform and thus unrelated to the boundary displacements that we detect or control in a typical experiment. From the experimental point of view, we do not know the local state of deformation in such circumstances and thus cannot be sure that the process is indeed cyclic. Of course, homogeneous deformations are directly related to boundary displacements, as discussed previously in the context of rubber bands. Again we digress.

Localize and we obtain the pointwise restriction

$$\int_{t_1}^{t_2} \hat{\mathbf{P}}_\kappa(\mathbf{F}) \cdot \dot{\mathbf{F}} dt \geq 0 \tag{3.26}$$

in the case of elasticity.

To explore the consequences of this inequality, let $\mathbf{F}(t)$ be the deformation gradient at the material point considered, associated with a cyclic process. Consider another process with gradient $\mathbf{F}^*(t)$ (not the cofactor), defined by $\mathbf{F}^*(t) = \mathbf{F}(\tau)$ with $\tau = t_1 + t_2 - t$. This is the simply the reversal of the original process; that is, $\mathbf{F}^*(t_{1,2}) = \mathbf{F}(t_{2,1})$, $\dot{\mathbf{F}}^*(t) = -\dot{\mathbf{F}}(\tau)$ and $\dot{\mathbf{F}}^*(t_{1,2}) = -\dot{\mathbf{F}}(t_{2,1})$. It is, therefore, a cyclic process, and, hence, subject to the work inequality:

$$0 \leq \int_{t_1}^{t_2} \hat{\mathbf{P}}_\kappa(\mathbf{F}^*) \cdot \dot{\mathbf{F}}^* dt = -\int_{t_1}^{t_2} \hat{\mathbf{P}}_\kappa(\mathbf{F}(\tau)) \cdot \dot{\mathbf{F}}(\tau) d\tau, \tag{3.27}$$

which is just eqn (3.26) with the inequality reversed. Therefore, for elasticity,

$$\int_{t_1}^{t_2} \hat{\mathbf{P}}_\kappa(\mathbf{F}) \cdot \dot{\mathbf{F}} dt = 0 \tag{3.28}$$

in a cyclic process.

Now, as t traverses the interval $[t_1, t_2]$, the deformation gradient traces out a curve in the nine-dimensional space Lin^+. Suppose C is such a curve, and suppose it is closed and smooth, so that it meets the conditions associated with a cyclic process. Then, eqn (3.28) is equivalent to

$$\oint_C \hat{\mathbf{P}}_\kappa(\bar{\mathbf{F}}) \cdot d\bar{\mathbf{F}} = 0. \tag{3.29}$$

Let $\mathbf{F}_0$ and $\mathbf{F}$ be distinct points on C, and let Γ_1 and Γ_2 be the two disjoint parts of C connecting $\mathbf{F}_0$ to $\mathbf{F}$. Then, eqn (3.29) may be expressed as

$$\int_{\Gamma_1} \hat{\mathbf{P}}_\kappa(\bar{\mathbf{F}}) \cdot d\bar{\mathbf{F}} = \int_{\Gamma_2} \hat{\mathbf{P}}_\kappa(\bar{\mathbf{F}}) \cdot d\bar{\mathbf{F}}, \tag{3.30}$$

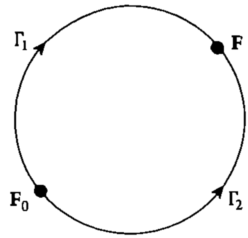

Figure 3.1 A cyclic process in deformation-gradient space

implying that the path integral $\int_\Gamma \hat{\mathbf{P}}_\kappa(\bar{\mathbf{F}}) \cdot d\bar{\mathbf{F}}$, where Γ is any smooth curve connecting $\mathbf{F}_0$ and $\mathbf{F}$, is in fact the same for all paths having the same endpoints and is thus dependent only on the latter (see Figure 3.1).

Fixing $\mathbf{F}_0$ we thus have a function

$$W(\mathbf{F}) = \int_\Gamma \hat{\mathbf{P}}_\kappa(\bar{\mathbf{F}}) \cdot d\bar{\mathbf{F}}, \tag{3.31}$$

modulo a constant. Let $\bar{\mathbf{F}}(\bar{u})$ be a parametrization of Γ, arranged such that $\mathbf{F}_0 = \bar{\mathbf{F}}(0)$ and $\mathbf{F} = \bar{\mathbf{F}}(u)$. Then, by elementary calculus,

$$W_{\mathbf{F}} \cdot \bar{\mathbf{F}}'(u) = W' = \hat{\mathbf{P}}_\kappa(\mathbf{F}) \cdot \bar{\mathbf{F}}'(u), \tag{3.32}$$

where the dash is an ordinary derivative with respect to u. This is the same as (3.10) and carries the same consequence; namely, the connection eqn (3.11).

Conversely, if eqn (3.11) holds then $\hat{\mathbf{P}}_\kappa(\bar{\mathbf{F}}) \cdot d\bar{\mathbf{F}} = dW(\bar{\mathbf{F}})$, ensuring that eqn (3.29) is satisfied. Thus, the work inequality for cyclic processes is satisfied by elastic materials if and only if they are hyperelastic.

Problems

1. If one observer concludes that an elastic material is hyperelastic, does every observer conclude the same? If so, how are their strain-energy functions related?
2. Prove the virtual-work theorem; i.e., show that a body is equilibrated *if and only if*

$$\int_\kappa \mathbf{P} \cdot \nabla \mathbf{v} dv = \int_\kappa \rho_\kappa \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial\kappa_p} \mathbf{p} \cdot \mathbf{v} da, \tag{3.33}$$

for all $\mathbf{v}$ that vanish on $\partial\kappa \setminus \partial\kappa_p$.

3. We showed that if an elastic material is hyperelastic; i.e., if $\hat{\mathbf{P}} = W_{\mathbf{F}}$, then the mechanical power theorem for the entire body may be expressed in the form $d\mathcal{E}/dt = \mathcal{P}$, where

$$\mathcal{E}(\kappa,t) = \mathcal{U}(\kappa,t) + \mathcal{K}(\kappa,t), \tag{3.34}$$

in which $\mathcal{K}$ is the kinetic energy, $\mathcal{P}$ is the power of the applied loads, and $\mathcal{U}$ is the strain energy. (Actually, we showed this for a sub-volume $\pi \subseteq \kappa$; the present special case is recovered on choosing $\pi = \kappa$.). Thus, the total mechanical energy $\mathcal{E}$ is conserved; i.e., it is independent of time, if there are no loads acting on the body. It is possible for non-zero applied forces to generate a *conservation law* of the same kind. These forces should be such such that $\mathcal{P} = d\mathcal{L}/dt$ for some function $\mathcal{L}$. In this case, the motion satisfies the conservation law $d\mathcal{E}'/dt = 0$, where $\mathcal{E}' = (\mathcal{U} - \mathcal{L}) + \mathcal{K}$. The term in parentheses is called the *potential energy* of the body and loads, in combination. Because of this conservation law, such forces are called *conservative*.

(a) Show that *dead loading*, in which $\mathbf{b}$ and $\mathbf{p}$ respectively are assigned as functions of $\mathbf{x}$ in κ and on $\partial\kappa_p$, furnishes an example of conservative loading. What is the load potential $\mathcal{L}$?

(b) State conditions under which a pressure load $\mathbf{t} = -p\mathbf{n}$ is conservative, where $\mathbf{t}$ is the Cauchy traction, p is the pressure, and $\mathbf{n}$ is the exterior unit normal to the surface of the body after deformation. Give the corresponding load potentials.

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4

Material symmetry

4.1 Stress response

Consider what happens if we surgically remove a small neighborhood of a material point in a reference configuration, subject it to some sort of transformation such as a deformation, and then re-insert it. If the material response to a given deformation, represented by the stress or the strain energy, should happen to be the same as it was before the operation, then the latter is mechanically undetectable and the two local neighborhoods—that before the operation and the one after—are effectively indistinguishable as far as the properties of the material are concerned. Such an operation is called a *material symmetry transformation.* Our purpose in this chapter is to outline the general theory of such transformations and to apply it to some practical examples.

Before doing this, it is necessary to understand how a change of reference manifests itself in the theory. This is the lesson of Problem 1 in Chapter 2. Thus, let κ and μ be two reference configurations and suppose, as before, that R is the configuration occupied by the body at time t. Then,

$$\chi_\kappa(\mathbf{x}, t) = \mathbf{y} = \chi_\mu(\mathbf{z}, t), \tag{4.1}$$

where $\mathbf{x}$ and $\mathbf{z}$, respectively, are the positions of material point p in κ and μ. Let $\lambda(\cdot)$ be the fixed map that transforms κ to μ; that is, $\mathbf{z} = \lambda(\mathbf{x})$. Because the two references are in one-to-one correspondence with points of the body, they are in such correspondence with each other as well. This means that λ is invertible. By the inverse function theorem, its gradient $\mathbf{R} = \nabla\lambda(\mathbf{x})$ is then an invertible tensor. Applying the chain rule to eqn (4.1) and reinstating the appropriate subscripts, we derive

$$\mathbf{F}_\kappa d\mathbf{x} = d\mathbf{y} = \mathbf{F}_\mu d\mathbf{z} = \mathbf{F}_\mu \mathbf{R} d\mathbf{x}, \quad \text{and hence} \quad \mathbf{F}_\kappa = \mathbf{F}_\mu \mathbf{R}. \tag{4.2}$$

For the time being we confine attention to Cauchy elasticity, returning to hyperelasticity later. Since the change of reference is merely a change in the way we record information, it has nothing to do with the actual state of the material at time t, which is thus unaffected by the change. Accordingly,

$$\mathbf{T}(p,t) = \mathbf{G}_\kappa(\mathbf{F}_\kappa;\mathbf{x}) = \mathbf{G}_\mu(\mathbf{F}_\mu;\mathbf{z}), \quad \text{where} \quad \mathbf{G}_\mu(\mathbf{F}_\mu;\mathbf{z}) = \mathbf{G}_\kappa(\mathbf{F}_\mu\mathbf{R}(\mathbf{z});\lambda^{-1}(\mathbf{z})). \tag{4.3}$$

This is how the constitutive *function* for the Cauchy stress is obtained when the reference configuration is changed.

Let us focus attention on a particular material point p_0. Because the stress at this point is sensitive only to deformations in some neighborhood of it, we need only consider local changes of reference $N_\kappa(p_0) \to N_\mu(p_0)$, say, where $N_\mu(p_0)$ is the image of $N_\kappa(p_0)$ under the map λ. This allows us to effectively marginalize the parametric dependence of the constitutive function on reference position, in the case of non-uniform materials, simply by arranging λ such that $\lambda(\mathbf{x}_0) = \mathbf{x}_0$, where $\mathbf{x}_0$ is the reference position of p_0; thus,

$$\mathbf{T}(p_0,t) = \mathbf{G}_\kappa(\mathbf{F}_\kappa;\mathbf{x}_0) = \mathbf{G}_\mu(\mathbf{F}_\mu;\mathbf{x}_0), \quad \text{where} \quad \mathbf{F}_\kappa = \mathbf{F}_\mu\mathbf{R}(\mathbf{x}_0). \tag{4.4}$$

Consider now an experiment in which $N_\kappa(p_0)$ is subjected to a deformation $\chi(\mathbf{x},t)$, for $\mathbf{x} \in N_\kappa(p_0)$; the response at p_0 is $\mathbf{G}_\kappa(\mathbf{F}_\kappa;\mathbf{x}_0)$, where $\mathbf{F}_\kappa$ is the gradient of $\chi(\mathbf{x},t)$ at $\mathbf{x}_0$. Let $N_\mu(p_0)$ be subjected to the same deformation; that is, to $\chi(\mathbf{z},t)$, for $\mathbf{z} \in N_\mu(p_0)$, in which $\chi(\cdot,t)$ is the same *function*. The response is $\mathbf{G}_\mu(\mathbf{F}_\mu;\mathbf{x}_0)$, where $\mathbf{F}_\mu$ is the gradient of $\chi(\mathbf{z},t)$ at $\mathbf{x}_0$. Note that $\mathbf{F}_\kappa = \mathbf{F}_\mu (= \mathbf{F}$, say) in this case, because they are the gradients of the same function, evaluated at the same point. So, the responses elicited by the experiment on $N_\kappa(p_0)$ and $N_\mu(p_0)$ are $\mathbf{G}_\kappa(\mathbf{F};\mathbf{x}_0)$ and $\mathbf{G}_\mu(\mathbf{F};\mathbf{x}_0)$, respectively. These need not have any relation to each other, except in the case when $\mathbf{z} = \lambda(\mathbf{x})$ is a symmetry transformation, in which case they coincide. That is, symmetry transformations at p_0 are such as to satisfy

$$\mathbf{G}_\kappa(\mathbf{F};\mathbf{x}_0) = \mathbf{G}_\mu(\mathbf{F};\mathbf{x}_0). \tag{4.5}$$

For, no experiment involving measurement of the Cauchy stress can then distinguish between $N_\kappa(p_0)$ and $N_\mu(p_0)$. Combining this with eqn (4.4), part 2, we derive

$$\mathbf{G}_\kappa(\mathbf{F};\mathbf{x}_0) = \mathbf{G}_\kappa(\mathbf{F}\mathbf{R};\mathbf{x}_0), \quad \text{where} \quad \mathbf{R} = \nabla\lambda(\mathbf{x}_0) \tag{4.6}$$

is the gradient at p_0 of the symmetry transformation, in which $\mathbf{F} \in Lin^+$ is arbitrary. Given the set of all such transformations, this amounts to a restriction on the *function* $\mathbf{G}_\kappa(\cdot;\mathbf{x}_0)$. Since it requires $\mathbf{FR}$ to be the pointwise value of a deformation gradient whenever $\mathbf{F}$ is, the restriction makes sense only if $\det\mathbf{R} > 0$. This, in turn, implies that symmetry transformations correspond to possible deformations of the material. We shall have reason to return to this point later.

The following observation is fundamental: Let $\mathcal{G}_{\kappa(p_0)}$ be the set of all $\mathbf{R}$ such that eqn (4.6) is satisfied (not to be confused with the response function (1.2)); that is,

$$\mathcal{G}_{\kappa(p_0)} = \left\{\mathbf{R}: \quad \mathbf{G}_\kappa(\mathbf{F};\mathbf{x}_0) = \mathbf{G}_\kappa(\mathbf{F}\mathbf{R};\mathbf{x}_0)\right\}. \tag{4.7}$$

Then this set is a *group*, in the sense that

$$
\begin{aligned}
&(i) \quad \mathbf{I} \in \mathcal{G}_{\kappa(p_0)},\\
&(ii) \quad \text{If} \quad \mathbf{R}_1, \mathbf{R}_2 \in \mathcal{G}_{\kappa(p_0)}, \quad \text{then} \quad \mathbf{R}_1\mathbf{R}_2 \in \mathcal{G}_{\kappa(p_0)},\\
&(iii) \quad \text{If} \quad \mathbf{R} \in \mathcal{G}_{\kappa(p_0)}, \quad \text{then} \quad \mathbf{R}^{-1} \in \mathcal{G}_{\kappa(p_0)}.
\end{aligned}
\tag{4.8}
$$

The first of these is obvious from the definition of $\mathcal{G}_{\kappa(p_0)}$. To prove the second, we observe (suppressing the passive argument $\mathbf{x}_0$) that $\mathbf{G}_\kappa(\mathbf{F}(\mathbf{R}_1\mathbf{R}_2)) = \mathbf{G}_\kappa((\mathbf{F}\mathbf{R}_1)\mathbf{R}_2) = \mathbf{G}_\kappa(\mathbf{F}\mathbf{R}_1) = \mathbf{G}_\kappa(\mathbf{F})$, and the third follows from $\mathbf{G}_\kappa(\mathbf{F}\mathbf{R}^{-1}) = \mathbf{G}_\kappa((\mathbf{F}\mathbf{R}^{-1})\mathbf{R}) = \mathbf{G}_\kappa(\mathbf{F}\mathbf{R}^{-1}\mathbf{R}) = \mathbf{G}_\kappa(\mathbf{F})$.

Note that $\mathbf{R} \in \mathcal{G}_{\kappa(p_0)}$ implies that $\mathbf{R}^n \in \mathcal{G}_{\kappa(p_0)}$ for any integer $n > 0$. Thus, $\mathbf{G}_\kappa(\mathbf{F}) = \mathbf{G}_\kappa(\mathbf{F}\mathbf{R}^n)$, where $\det(\mathbf{F}\mathbf{R}^n) = (\det\mathbf{F})(\det\mathbf{R}^n) = (\det\mathbf{F})(\det\mathbf{R})^n$. Let $n \to \infty$. Then, if $\det\mathbf{R} > 1$ we have $\det(\mathbf{F}\mathbf{R}^n) \to \infty$, corresponding to unbounded dilation; whereas, if $\det\mathbf{R} < 1$ we have $\det(\mathbf{F}\mathbf{R}^n) \to 0$, corresponding to unbounded compaction. Material symmetry then requires that the stress remain unaffected by unlimited dilation or compaction of the material. This is plainly unphysical, and so we impose the requirement

$$
\mathcal{G}_{\kappa(p_0)} \subseteq U = \{\mathbf{R}: \quad \det\mathbf{R} = 1\}. \tag{4.9}
$$

U is called the *unimodular group*.

Noll introduced the elegant idea of characterizing elastic materials as either fluid or solid, depending on the nature of the symmetry group. For example, in an inviscid compressible fluid the Cauchy stress is given by

$$
\mathbf{T} = -p(\rho)\mathbf{I}, \tag{4.10}
$$

where $p(\rho)$ is the pressure–density relation. In this case, we have $\mathbf{G}_\kappa(\mathbf{F};\mathbf{x}) = -p(\rho_\kappa(\mathbf{x})/\det\mathbf{F})\mathbf{I}$, yielding $\mathbf{G}_\kappa(\mathbf{F}\mathbf{R};\mathbf{x}) = -p(\rho_\kappa(\mathbf{x})/\det(\mathbf{F}\mathbf{R}))\mathbf{I}$. It follows immediately that $\mathcal{G}_{\kappa(p_0)} = U$ and, so in view of eqn (4.9), we are justified in saying that fluids have maximal symmetry.

For solids we assume the existence of $N_\kappa(p)$ such that

$$
\mathcal{G}_{\kappa(p)} \subseteq Orth^+. \tag{4.11}
$$

Such $N_\kappa(p)$ is called a local *undistorted* configuration. The idea is motivated by the structure of a unit cell of an undistorted crystal lattice; these are mapped to themselves by discrete rotations. Furthermore, we have in mind the fact that, for solids, a change in shape is detectable by experiment. Accordingly, the map λ is detectable if it generates a strain. Symmetry transformations should, therefore, be strain-free, and this, in turn, implies that $\mathbf{R}^t\mathbf{R} = \mathbf{I}$. The restriction eqn (4.9) then yields eqn (4.11), even for non-crystalline solids. Isotropic solids are those for which $N_\kappa(p)$ exists such that

$$
\mathcal{G}_{\kappa(p)} = Orth^+. \tag{4.12}
$$

Note that we have not invoked frame invariance. For constitutive functions that are admissible from this standpoint we use eqn (2.19) to conclude that

$$
\mathbf{G}_\kappa(\mathbf{F}\mathbf{R}) = \mathbf{F}[\mathbf{R}\mathbf{H}_\kappa(\mathbf{R}^t\mathbf{C}\mathbf{R})\mathbf{R}^t]\mathbf{F}^t, \tag{4.13}
$$

so that if $\mathbf{R} \in \mathcal{G}_{\kappa(p)}$, then

$$\mathbf{H}_\kappa(\mathbf{C}) = \mathbf{R}\mathbf{H}_\kappa(\mathbf{R}^t\mathbf{C}\mathbf{R})\mathbf{R}^t, \tag{4.14}$$

and if $\mathcal{G}_{\kappa(p)} \subseteq Orth^+$,

$$\mathbf{H}_\kappa(\mathbf{R}^t\mathbf{C}\mathbf{R}) = \mathbf{R}^t\mathbf{H}_\kappa(\mathbf{C})\mathbf{R}. \tag{4.15}$$

Suppose, for example, that a particular crystal lattice is such that the 180° rotation

$$\mathbf{R} = 2\mathbf{n} \otimes \mathbf{n} - \mathbf{I} \tag{4.16}$$

about the unit vector $\mathbf{n}$ belongs to $\mathcal{G}_{\kappa(p)}$. Then, both $\mathbf{R}$ and $-\mathbf{R}$ satisfy eqn (4.15), implying that the reflection through the plane with normal $\mathbf{n}$ is mechanically undetectable. In this way, the symmetry group may be extended to accommodate reflection symmetry with respect to crystallographic planes, despite the fact that such transformations cannot be associated with an actual deformation.

We have indicated that the symmetry group depends not only on the material, but also on the reference. To see precisely how this occurs consider a general invertible map π, with gradient $\mathbf{P}$ (not to be confused with the Piola stress) that takes reference κ_1, say, to κ_2. From eqn (4.3), part 2, we have the connection

$$\mathbf{G}_{\kappa_2}(\mathbf{F}; \mathbf{x}_2) = \mathbf{G}_{\kappa_1}(\mathbf{F}\mathbf{P}(\mathbf{x}_2); \pi^{-1}(\mathbf{x}_2)), \tag{4.17}$$

where $\mathbf{x}_2 = \pi(\mathbf{x}_1)$, implying that

$$\mathbf{G}_{\kappa_1}(\mathbf{F}; \pi^{-1}(\mathbf{x}_2)) = \mathbf{G}_{\kappa_2}(\mathbf{F}\mathbf{P}(\mathbf{x}_2)^{-1}; \mathbf{x}_2). \tag{4.18}$$

Suppose now that $\mathbf{R} \in \mathcal{G}_{\kappa_1(p)}$. Then,

$$\mathbf{G}_{\kappa_1}(\mathbf{F}; \pi^{-1}(\mathbf{x}_2)) = \mathbf{G}_{\kappa_1}(\mathbf{F}\mathbf{R}; \pi^{-1}(\mathbf{x}_2)). \tag{4.19}$$

Eqn (4.18), however, implies that

$$\mathbf{G}_{\kappa_1}(\mathbf{F}\mathbf{R}; \pi^{-1}(\mathbf{x}_2)) = \mathbf{G}_{\kappa_2}(\mathbf{F}\mathbf{R}\mathbf{P}(\mathbf{x}_2)^{-1}; \mathbf{x}_2), \tag{4.20}$$

and therefore (4.18) and (4.19) give

$$\mathbf{G}_{\kappa_2}(\mathbf{F}\mathbf{R}\mathbf{P}(\mathbf{x}_2)^{-1}; \mathbf{x}_2) = \mathbf{G}_{\kappa_1}(\mathbf{F}; \pi^{-1}(\mathbf{x}_2)) = \mathbf{G}_{\kappa_2}(\mathbf{F}\mathbf{P}(\mathbf{x}_2)^{-1}; \mathbf{x}_2). \tag{4.21}$$

Defining $\hat{\mathbf{F}} = \mathbf{F}\mathbf{P}^{-1}$, we recast this in the form

$$\mathbf{G}_{\kappa_2}(\hat{\mathbf{F}}; \mathbf{x}_2) = \mathbf{G}_{\kappa_2}(\hat{\mathbf{F}}\mathbf{P}\mathbf{R}\mathbf{P}^{-1}; \mathbf{x}_2), \tag{4.22}$$

and conclude that

$$\mathcal{G}_{\kappa_2(p)} = \left\{\mathbf{P}\mathbf{R}\mathbf{P}^{-1}: \quad \mathbf{R} \in \mathcal{G}_{\kappa_1(p)}\right\}. \tag{4.23}$$

This result is known as Noll's Rule.

It follows that if $\mathcal{G}_{\kappa_1(p)}$ satisfies eqn (4.11), then in general $\mathcal{G}_{\kappa_2(p)}$ does not. Therefore, the existence of a (local) reference configuration relative to which eqn (4.11) holds is an essential aspect of the definition of a solid. However, eqn (4.9) is always satisfied. In particular, fluids have the property that $\mathcal{G}_{\kappa(p)} = U$ for all choices of κ. This fact gives meaning to imprecise, but often-heard remarks to the effect that "fluids have no reference configuration." It would be better in this case to say that the structure of the constitutive function is insensitive to the reference configuration.

4.2 Strain energy

We have outlined the theory of material symmetry in terms of restrictions on the stress response. We may just as easily do so for the strain–energy response. Repeating the foregoing essentially verbatim, we arrive at the definition

$$g_{\kappa(p)} = \left\{\mathbf{R}: \quad W_\kappa(\mathbf{FR}; \mathbf{x}) = W_\kappa(\mathbf{F}; \mathbf{x})\right\} \tag{4.24}$$

of the associated symmetry group, which as before, is restricted by

$$g_{\kappa(p)} \subseteq U. \tag{4.25}$$

The obvious question is: How are $g_{\kappa(p)}$ and $\mathcal{G}_{\kappa(p)}$ related? To explore this, suppose $\mathbf{R} \in \mathcal{G}_{\kappa(p)}$. Then, using eqns (2.22) and (4.6) and suppressing the passive variable $\mathbf{x}$, it follows that

$$\hat{\mathbf{P}}_\kappa(\mathbf{FR}) = \mathbf{G}_\kappa(\mathbf{FR})(\mathbf{FR})^* = \hat{\mathbf{P}}_\kappa(\mathbf{F})\mathbf{R}^*, \tag{4.26}$$

where we have used the general rule $(\mathbf{AB})^* = \mathbf{A}^*\mathbf{B}^*$.

Problem

Prove this rule.

Thus, $\hat{\mathbf{P}}_\kappa(\mathbf{F}) = \hat{\mathbf{P}}_\kappa(\mathbf{FR})\mathbf{R}^t$, or

$$W_{\mathbf{F}}(\mathbf{F}) = W_{\bar{\mathbf{F}}}(\bar{\mathbf{F}})\mathbf{R}^t = W_{\mathbf{F}}(\bar{\mathbf{F}}), \quad \text{where} \quad \bar{\mathbf{F}} = \mathbf{FR}, \tag{4.27}$$

in which the chain rule has been used to derive the second equality and the subscript κ has been suppressed for clarity. Integration at fixed $\mathbf{R}$ then furnishes

$$W_\kappa(\mathbf{F}) = W_\kappa(\mathbf{FR}) + C(\mathbf{R}), \tag{4.28}$$

in which C is independent of $\mathbf{F}$. Evaluating at $\mathbf{F} = \mathbf{I}$ leads to the implication

$$W_\kappa(\mathbf{F}) = W_\kappa(\mathbf{FR}) + W_\kappa(\mathbf{I}) - W_\kappa(\mathbf{R}). \tag{4.29}$$

Conversely, if this is satisfied then by reversing the steps, we see that $\mathbf{R} \in \mathcal{G}_{\kappa(p)}$ and, thus, that the two statements are equivalent.

Next, suppose $\mathbf{R} \in g_{\kappa(p)}$, so that $W_\kappa(\mathbf{I}) = W_\kappa(\mathbf{R})$ in particular. Then eqn (4.29) is satisfied and so $\mathbf{R} \in \mathcal{G}_{\kappa(p)}$; that is, $g_{\kappa(p)} \subseteq \mathcal{G}_{\kappa(p)}$. Recall that admissible strain-energy functions meet the invariance requirement, eqn (3.15); in particular, $W_\kappa(\mathbf{Q}) = W_\kappa(\mathbf{I})$ for all rotations $\mathbf{Q}$. This means that for solids, i.e., for $\mathcal{G}_{\kappa(p)} \subseteq Orth^+$, any $\mathbf{R} \in \mathcal{G}_{\kappa(p)}$ satisfies $W_\kappa(\mathbf{R}) = W_\kappa(\mathbf{I})$, but such $\mathbf{R}$ also satisfies eqn (4.29), so that $W_\kappa(\mathbf{F}) = W_\kappa(\mathbf{FR})$ and $\mathbf{R} \in g_{\kappa(p)}$. We have thus shown, for solids, that $\mathcal{G}_{\kappa(p)} \subseteq g_{\kappa(p)}$ and, hence, that

$$g_{\kappa(p)} = \mathcal{G}_{\kappa(p)}. \tag{4.30}$$

The symmetry groups for stress and energy are thus one and the same. Beyond this, we may proceed exactly as in the case of the stress–response function to extend the symmetry group to include improper orthogonal transformations as needed to incorporate reflection symmetry, using the fact that admissible strain–energy functions are expressible as $W_\kappa(\mathbf{F}) = \hat{W}_\kappa(\mathbf{C})$, and the consequent fact that $\hat{W}_\kappa(\mathbf{C}) = \hat{W}_\kappa(\mathbf{R}^t\mathbf{CR})$ for all $\mathbf{R} \in g_{\kappa(p)}$. Of course, an explicit dependence on $\mathbf{x}$ is allowed, to cover non-homogeneous materials.

4.3 Isotropy

In view of the foregoing result, it is enough to characterize symmetry in terms of the strain–energy function. In the case of isotropy, then, there is presumed to exist $\kappa(p)$ such that $g_{\kappa(p)} = Orth^+$; that is,

$$\hat{W}_\kappa(\mathbf{C};\mathbf{x}) = \hat{W}_\kappa(\mathbf{R}^t\mathbf{CR};\mathbf{x}) \quad \text{for all} \quad \mathbf{R} \in Orth. \tag{4.31}$$

In virtually every text on continuum mechanics, it is established that this restriction is equivalent to the statement

$$\hat{W}_\kappa(\mathbf{C};\mathbf{x}) = U(I_1, I_2, I_3;\mathbf{x}), \tag{4.32}$$

for some function U, where

$$I_1 = tr\mathbf{C}, \quad I_2 = tr\mathbf{C}^* = \frac{1}{2}[I_1^2 - tr(\mathbf{C}^2)] \quad \text{and} \quad I_3 = \det\mathbf{C} \tag{4.33}$$

are the principal invariants of $\mathbf{C}$. The proof is a model for extensions to other kinds of symmetry, such as that described in the next chapter, and so, at the risk of being repetitive, we pause to outline it explicitly.

Suppose, then, that $\mathbf{A},\mathbf{B} \in Sym^{+}$, the set of positive-definite symmetric tensors, and that these are such that their invariants coincide: $I_k(\mathbf{A}) = I_k(\mathbf{B})$. Then, because the invariants define the characteristic equation

$$\mu^3 - I_1\mu^2 + I_2\mu - I_3 = 0 \tag{4.34}$$

for the eigenvalues μ, it follows that $\mathbf{A}$ and $\mathbf{B}$ also share the same (real-valued and positive) eigenvalues. From the spectral representation for symmetric tensors it is concluded that

$$\mathbf{A} = \sum \mu_i \mathbf{a}_i \otimes \mathbf{a}_i \quad \text{and} \quad \mathbf{B} = \sum \mu_i \mathbf{b}_i \otimes \mathbf{b}_i, \tag{4.35}$$

where the sets $\{\mathbf{a}_i\}$ and $\{\mathbf{b}_i\}$ are orthonormal. The latter property means that the tensor $\mathbf{Q}$, defined by $\mathbf{Q} = \mathbf{a}_i \otimes \mathbf{b}_i$, is orthogonal. Thus,

$$\mathbf{B} = \sum \mu_i \mathbf{Q}^t\mathbf{a}_i \otimes \mathbf{Q}^t\mathbf{a}_i = \mathbf{Q}^t \left(\sum \mu_i \mathbf{a}_i \otimes \mathbf{a}_i\right) \mathbf{Q} = \mathbf{Q}^t\mathbf{A}\mathbf{Q}, \tag{4.36}$$

and eqn (4.31) implies that $\hat{W}_\kappa(\mathbf{A};\mathbf{x}) = \hat{W}_\kappa(\mathbf{Q}^t\mathbf{A}\mathbf{Q};\mathbf{x}) = \hat{W}_\kappa(\mathbf{B};\mathbf{x})$, meaning that every $\hat{W}_\kappa(\cdot;\mathbf{x})$ satisfying eqn (4.31) is determined by the principal invariants of its argument and, hence, that eqn (4.32) is valid. Conversely, if the latter is true, then since $I_k(\mathbf{R}^t\mathbf{C}\mathbf{R}) = I_k(\mathbf{C})$ for all orthogonal $\mathbf{R}$, eqn (4.31) follows, and is thus necessary and sufficient for eqn (4.32).

To obtain the stress we use the chain rule in the form

$$Sym\hat{W}_{\mathbf{C}} \cdot \dot{\mathbf{C}} = (\hat{W})^{\cdot} = \sum_{k=1}^{3} U_k \dot{I}_k = \sum_{k=1}^{3} U_k Sym(I_k)_{\mathbf{C}} \cdot \dot{\mathbf{C}}, \tag{4.37}$$

where $U_k = \partial U/\partial I_k$. Using (Supplemental Notes, Part 4)

$$Sym(I_1)_{\mathbf{C}} = \mathbf{I}, \quad Sym(I_2)_{\mathbf{C}} = I_1\mathbf{I} - \mathbf{C} \quad \text{and} \quad Sym(I_3)_{\mathbf{C}} = \mathbf{C}^* = I_3\mathbf{C}^{-1}, \tag{4.38}$$

together with eqns (2.22), (3.11), and (3.20), we thus derive

$$J\mathbf{T} = 2[(U_1 + I_1U_2)\mathbf{B} - U_2\mathbf{B}^2 + I_3U_3\mathbf{I}], \tag{4.39}$$

where $J = \sqrt{I_3}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^t$ is the left Cauchy–Green deformation tensor.

In the next chapter, the foregoing argument is extended to obtain the general form of the response function for transversely isotropic materials. For these, there exists κ such that

$$W_\kappa(\mathbf{F};\mathbf{x}) = W_\kappa(\mathbf{FR};\mathbf{x}) \quad \text{for all} \quad \mathbf{R} \in Orth^+ \quad \text{with axis } \mathbf{m}; \tag{4.40}$$

that is, for all rotations $\mathbf{R}$ such that $\mathbf{Rm} = \mathbf{m}$, the axis of transverse isotropy.

However, it is time for some exercises.

Problems 1

1. Prove that U is, indeed, a group.
2. Prove the second equality in eqn (4.27).
3. Let $W_\kappa(\mathbf{F};\mathbf{x}) = w(J;\mathbf{x})$, where $-dw/dJ = p(\rho)$, the pressure–density relation for fluids. Show that $g_{\kappa(p)} = U$ and, hence, that eqn (4.30) is satisfied for fluids.
4. We established, for solids, that the symmetry group for admissible stress–response functions may be extended to include elements of *Orth*, the set of orthogonal tensors. Show that the same conclusion applies to the symmetry group based on energy response.
5. Given eqn (4.39) for some choice of reference, compute the stress–response function relative to any other choice.
6. Show that, for an isotropic elastic material, the strain–energy function is expressible as a function of the principal stretches, i.e., $U = \omega(\lambda_1, \lambda_2, \lambda_3; \mathbf{x})$, in which the stretches can be permuted arbitrarily (i.e., they can appear in this function in any order). Use $\dot{W} = \frac{1}{2}\mathbf{S}\cdot\dot{\mathbf{C}}$ and the spectral representation $\mathbf{C} = \sum \lambda_i^2 \mathbf{u}_i \otimes \mathbf{u}_i$ (note that $|\mathbf{u}_i| = 1$, but $\dot{\mathbf{u}}_i \neq \mathbf{0}$!) to obtain the representation

$$\mathbf{P} = \sum \frac{\partial \omega}{\partial \lambda_i}\mathbf{v}_i \otimes \mathbf{u}_i, \tag{4.41}$$

   where $\mathbf{v}_i = \mathbf{Ru}_i$ and $\mathbf{R}$ is the rotation in the polar decomposition of $\mathbf{F}$. Thus, $\mathbf{U} = \sum \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i$, $\mathbf{R} = \mathbf{v}_i \otimes \mathbf{u}_i$ and $\mathbf{F} = \sum \lambda_i \mathbf{v}_i \otimes \mathbf{u}_i$. Show that the cofactor of $\mathbf{F}$ may be represented in the form $\mathbf{F}^* = \sum \mu_i \mathbf{v}_i \otimes \mathbf{u}_i$, where $\mu_i = J/\lambda_i$. All sums range over $\{1, 2, 3\}$.
7. For an isotropic (relative to κ) material we have $W(\mathbf{F}(u)) = W(\mathbf{F})$ where $\mathbf{F}(u) = \mathbf{FQ}(u)$, $\mathbf{F}$ is fixed and $\mathbf{Q}(u)$ is a one-parameter family of rotations with $\mathbf{Q}(0) = \mathbf{I}$. Thus, $W' = 0$, where $W' = dW/du$. Prove that $\mathbf{F}^t\mathbf{P}$ must then be symmetric and that this, in turn, is equivalent to the statement $\mathbf{TB} = \mathbf{BT}$, granted the symmetry of the Cauchy stress. Reverse the argument and show that $\mathbf{TB} = \mathbf{BT}$ implies isotropy. This condition is therefore necessary and sufficient for isotropy, granted the symmetry of the Cauchy stress. Show that the condition is equivalent to three *universal relations* that apply to all isotropic materials:

$$\begin{aligned} B_{12}(T_{11} - T_{22}) &= T_{12}(B_{11} - B_{22}) + T_{32}B_{13} - T_{13}B_{32} \\ B_{23}(T_{22} - T_{33}) &= T_{23}(B_{22} - B_{33}) + T_{13}B_{21} - T_{21}B_{13} \\ B_{31}(T_{33} - T_{11}) &= T_{31}(B_{33} - B_{11}) + T_{21}B_{32} - T_{32}B_{21}, \end{aligned} \tag{4.42}$$

   where the components are referred to $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$.

Consider a unit cube subjected to the homogeneous deformation

$$y_1 = a_1x_1 + ka_2x_2, \quad y_2 = a_2x_2, \quad y_3 = a_3x_3, \tag{4.43}$$

where a_1, a_2, a_3, and k are positive constants. Note that for homogeneous materials this deformation is automatically in equilibrium in the absence of body forces.

Show that the universal relations yield a single relation of the form

$$T_{11} - T_{22} = T_{12}F(a_1, a_2, k). \tag{4.44}$$

Find expressions for the traction, $\mathbf{t}$, acting on the material planes $x_1 = const.$ and $x_2 = const.$, and suppose $\mathbf{n} \cdot \mathbf{t}$ vanishes on these planes in the deformed configuration, so that only shear tractions are acting. Obtain the purely geometric relation

$$a_1^2 = (1 + k^2)a_2^2. \tag{4.45}$$

This furnishes a simple necessary condition for isotropy that can be tested experimentally. Indicate the meaning of this equation on a figure.

Before leaving this discussion we pause to develop a formulation of isotropic elasticity that has proved to be convenient in applications. To this end, we use the result of Problem no. 6 above, together with

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2 \quad \text{and} \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2, \tag{4.46}$$

where the λ_i are the principal stretches. These, of course, are the principal invariants of $\mathbf{C}$. The principal invariants of $\mathbf{U}$, namely

$$i_1 = tr\mathbf{U}, \quad i_2 = tr\mathbf{U}^* \quad \text{and} \quad i_3 = \det \mathbf{U}, \tag{4.47}$$

are

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \quad \text{and} \quad i_3 = \lambda_1\lambda_2\lambda_3, \tag{4.48}$$

which may be used with (4.46) to verify that

$$I_1 = i_1^2 - 2i_2, \quad I_2 = i_2^2 - 2i_1i_3 \quad \text{and} \quad I_3 = i_3^2. \tag{4.49}$$

It then follows from (4.32) that the strain-energy function, in the case of isotropy, is expressible in the form

$$W_\kappa(\mathbf{C}; \mathbf{x}) = w(i_1, i_2, i_3; \mathbf{x}), \quad \text{where} \quad w(i_1, i_2, i_3; \mathbf{x}) = U(i_1^2 - 2i_2, i_2^2 - 2i_1i_3, i_3^2; \mathbf{x}). \tag{4.50}$$

As a useful corollary, the strain–energy is a completely symmetric function ω of the principal stretches in this case, remaining invariant with respect to interchange of any two of them. In fact, the nonlinear system eqn (4.49) is uniquely invertible. Its inverse may be expressed in the form eqn (4.48), in which (see Rivlin's (2004) paper)

$$\lambda_i = \frac{1}{\sqrt{3}}\left\{I_1 + 2A\cos\left[\frac{1}{3}(\phi - 2\pi i)\right]\right\}^{1/2}; \quad i = 1, 2, 3, \tag{4.51}$$

where

$$A = (I_1^2 - 3I_2)^{1/2} \quad \text{and} \quad \phi = \cos^{-1}\left[\frac{1}{2A^3}(2I_1^3 - 9I_1I_2 + 27I_3)\right]. \tag{4.52}$$

A simple application of the chain rule yields

$$\partial\omega/\partial\lambda_1 = w_1 + (\lambda_2 + \lambda_3)w_2 + \lambda_2\lambda_3 w_3, \tag{4.53}$$

where $w_k = \partial w/\partial i_k$, together with two similar relations obtained by permuting the principal stretches. Using these relations in the solution to Problem no. 6 above, we derive

$$\begin{aligned} W_{\mathrm{F}} = w_1 \sum \mathbf{v}_i \otimes \mathbf{u}_i &+ w_2[(\lambda_2 + \lambda_3)\mathbf{v}_1 \otimes \mathbf{u}_1 + (\lambda_1 + \lambda_3)\mathbf{v}_2 \otimes \mathbf{u}_2 \\ &+ (\lambda_2 + \lambda_2)\mathbf{v}_3 \otimes \mathbf{u}_3] + w_3(\lambda_2\lambda_3\mathbf{v}_1 \otimes \mathbf{u}_1 + \lambda_1\lambda_3\mathbf{v}_2 \otimes \mathbf{u}_2 + \lambda_1\lambda_2\mathbf{v}_3 \otimes \mathbf{u}_3). \end{aligned} \tag{4.54}$$

The first sum on the right-hand side is recognizable as $\mathbf{R}$, the rotation factor in the polar decomposition $\mathbf{F} = \mathbf{RU}$ of $\mathbf{F}$; the third is just $\mathbf{RU}^*(= \mathbf{R}^*\mathbf{U}^* = \mathbf{F}^*)$; and the second is

$$(\lambda_2 + \lambda_3)\mathbf{v}_1 \otimes \mathbf{u}_1 + \ldots = (\lambda_1 + \lambda_2 + \lambda_3)\sum \mathbf{v}_i \otimes \mathbf{u}_i - \sum \lambda_i \mathbf{v}_i \otimes \mathbf{u}_i = i_1\mathbf{R} - \mathbf{F}. \tag{4.55}$$

Thus,

$$\mathbf{P} = W_{\mathrm{F}} = \mathbf{R}\sigma, \tag{4.56}$$

where

$$\sigma = (w_1 + i_1 w_2)\mathbf{I} - w_2\mathbf{U} + w_3\mathbf{U}^* \tag{4.57}$$

is the Biot stress tensor.

We caution the reader that the factorization eqn (4.56) of the Piola stress into the rotation and the Biot stress is appropriate only in the case of isotropy, whereas a tensor usually referred to as the Biot stress, which yields eqns (4.56) and (4.57) in the case of isotropy, has a wider significance. We shall not need the general form here, however, and so suggest that reference be made to Ogden (1997) for further discussion.

Problems

1. Use the relations between the invariants I_k of $\mathbf{C}$ (or $\mathbf{B}$) and the invariants i_k of $\mathbf{U}$ (or $\mathbf{V}$) discussed previously to establish the three-dimensional formula

$$i\mathbf{R} = i_1\mathbf{F}^* - \mathbf{FC} + (i_1^2 - i_2)\mathbf{F}, \quad \text{where} \quad i = i_1 i_2 - i_3 = (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3). \tag{4.58}$$

   In practice, one has direct access to the I_k from, say, a finite-element analysis in which the deformation gradient is obtained from nodal displacement data. Discuss the problem of obtaining the i_k in terms of the I_k and, thus, obtaining $\mathbf{R}$ in terms of the deformation gradient directly. You should appreciate the convenience afforded by this method because it means one does not have to go through all the steps needed to carry out a computationally intensive polar decomposition in applications calling for the evaluation of $\mathbf{R}$.

2. Establish the formulas

$$(i_1)_\mathbf{F} = \mathbf{R}, \quad (i_2)_\mathbf{F} = i_1\mathbf{R} - \mathbf{F}, \quad (i_3)_\mathbf{F} = \mathbf{F}^* \quad \text{and} \quad (i_2)_{\mathbf{F}^*} = \mathbf{R}. \tag{4.59}$$

At this stage, it is instructive to revisit eqn (2.15). Recall that this is formally equivalent to the statement that the constitutive function adopted by observer $\mathcal{O}$ is insensitive to arbitrary rigid-body motions superposed on a given motion. In fact, it is widespread practice to impose this requirement in place of frame invariance. However, this interpretation of eqn (2.15) is flawed, if only because it is not possible to subject a deformable body to an arbitrary rigid-body motion. To see this, imagine a uniform, isotropic elastic body undergoing the rigid-body motion

$$\mathbf{y} = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t), \tag{4.60}$$

with $\mathbf{Q} \in Orth^+$. Then, $\mathbf{F} = \mathbf{Q}$, yielding $\mathbf{R} = \mathbf{Q}$ and $\mathbf{U} = \mathbf{I}$, and the Cauchy stress $\mathbf{T}$, defined by $\mathbf{P} = \mathbf{TF}^*$ reduces, with the aid of eqn (4.39), to

$$\mathbf{T} = c\mathbf{I}, \tag{4.61}$$

where c is a constant. The divergence of the Cauchy stress vanishes. If no body forces are acting, the equation of motion (2.12) reduces to

$$\mathbf{0} = \mathbf{A}(t)\mathbf{y} + \mathbf{d}(t), \tag{4.62}$$

where $\mathbf{A} = \ddot{\mathbf{Q}}\mathbf{Q}^\prime$ and $\mathbf{d} = \ddot{\mathbf{c}} - \mathbf{Ac}$. Evaluating the gradient with respect to $\mathbf{y}$ at an arbitrary point of the body yields $\mathbf{A} = \mathbf{0}$; thus, $\ddot{\mathbf{Q}} = \mathbf{0}$, and $\ddot{\mathbf{c}} = \mathbf{0}$. If we identify the reference configuration with the initial configuration of the body and assume the initial velocity to vanish pointwise, then the initial value of $\mathbf{Q}$ is $\mathbf{I}$, and the initial values of $\dot{\mathbf{Q}}$, $\mathbf{c}$ and $\dot{\mathbf{c}}$ all vanish, yielding $\mathbf{y} = \mathbf{x}$ for all t. The only rigid motion is then the trivial motion in which the body remains stationary.

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5

Fiber symmetry

The term *fiber symmetry* refers to a symmetry group consisting of rotations about an axis. Typically, this axis is identified with the unit tangent to a fiber embedded in the material, as in a fiber-reinforced composite or a fibrous biological tissue (Figure 5.1). Such materials are said to be *transversely isotropic*; they are effectively isotropic in the plane orthogonal to the fiber direction. Our objective here is to solve the representation problem for transversely isotropic strain–energy functions, i.e., to find the maximal list of variables upon which these functions may depend. We have already solved the representation problem for isotropy, concluding, in that case, that the energy is a general function of the principal invariants $I_{1,2,3}$ of $\mathbf{C}$.

Recall that the general restriction imposed by material symmetry is

$$\hat{W}(\mathbf{C}) = \hat{W}(\mathbf{R}^t\mathbf{C}\mathbf{R}) \tag{5.1}$$

for all positive definite, symmetric $\mathbf{C}$, and for all $\mathbf{R} \in g_{\kappa(p)}$, the symmetry group relative to configuration κ at the material point p. This (local) configuration is *undistorted* if $g_{\kappa(p)} \subset$ *Orth*, the group of orthogonal tensors.

Transverse isotropy is associated with the symmetry group

$$g_{\kappa(p)} = \left\{\mathbf{Q}: \quad \mathbf{Q} \in Orth \quad \text{and} \quad \mathbf{Q}\mathbf{m}(\mathbf{x}) = \pm\mathbf{m}(\mathbf{x}) \quad \text{with} \quad |\mathbf{m}(\mathbf{x})| = 1\right\}, \tag{5.2}$$

where $\mathbf{m}(\mathbf{x})$ is the fiber axis at the material point p. As all arguments presented here are purely local, henceforth, we suppress this material point in the notation. The strain energy is thus invariant under all rotations about the fiber axis, and under reflection through the plane—the isotropic plane—perpendicular to this axis.

As a prelude to our main theorem, note that if $\mathbf{Q}\mathbf{m} = \pm\mathbf{m}$, then, as $\mathbf{Q} \in Orth$, we have $\mathbf{Q}^t\mathbf{m} = \pm\mathbf{m}$; this follows simply on multiplying by $\mathbf{Q}^t$. Thus, $\mathbf{R}^t \in g$, if $\mathbf{R} \in g$ (we drop the subscript on $g_{\kappa(p)}$). Moreover,

$$g = \{\mathbf{Q}: \quad \mathbf{Q} \in Orth \quad \text{and} \quad \mathbf{Q}\mathbf{M}\mathbf{Q}^t = \mathbf{M}\}, \tag{5.3}$$

Finite Elasticity Theory. David J. Steigmann.

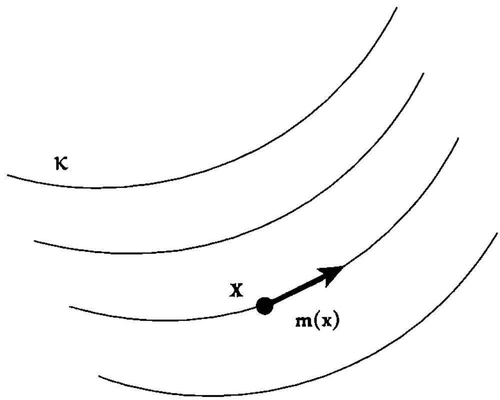

Figure 5.1 A material with a continuous distribution of embedded fibers

where

$$\mathbf{M} = \mathbf{m} \otimes \mathbf{m}. \tag{5.4}$$

This is called the *structural tensor* for transverse isotropy.

Proof First, we note that for either choice of sign,

$$\mathbf{m} \otimes \mathbf{m} = \pm\mathbf{m} \otimes \pm\mathbf{m} = \mathbf{Qm} \otimes \mathbf{Qm} = \mathbf{Q}(\mathbf{m} \otimes \mathbf{m})\mathbf{Q}^t, \tag{5.5}$$

and so $\mathbf{M} = \mathbf{QMQ}^t$. This shows that $\mathbf{Qm} = \pm\mathbf{m} \Rightarrow \mathbf{QMQ}^t = \mathbf{M}$. To show the converse, suppose $\mathbf{QMQ}^t = \mathbf{M}$ with $\mathbf{Q} \in Orth$ and $\mathbf{M}$ as defined in eqn (5.4). Then, $\mathbf{QM} = \mathbf{MQ}$, or $\mathbf{Q}(\mathbf{m} \otimes \mathbf{m}) = (\mathbf{m} \otimes \mathbf{m})\mathbf{Q}$, i.e., $\mathbf{Qm} \otimes \mathbf{m} = \mathbf{m} \otimes \mathbf{Q}^t\mathbf{m}$. Then,

$$\mathbf{Qm} = (\mathbf{Qm} \otimes \mathbf{m})\mathbf{m} = (\mathbf{m} \otimes \mathbf{Q}^t\mathbf{m})\mathbf{m} = (\mathbf{m} \cdot \mathbf{Qm})\mathbf{m}. \tag{5.6}$$

This, and the orthogonality of $\mathbf{Q}$, imply that $\mathbf{m} \cdot \mathbf{m} = \mathbf{Qm} \cdot \mathbf{Qm} = (\mathbf{m} \cdot \mathbf{Qm})^2\mathbf{m} \cdot \mathbf{m}$ and, hence, that $\mathbf{m} \cdot \mathbf{Qm} = \pm 1$. Thus, $\mathbf{QMQ}^t = \mathbf{M} \Leftrightarrow \mathbf{Qm} = \pm\mathbf{m}$.

Our strategy is to replace the representation problem eqn (5.1), eqn (5.2) by an equivalent representation problem for *isotropic* functions. We make use of the following:

Theorem *$\hat{W}$ is invariant under g, i.e., $\hat{W}(\mathbf{C}) = \hat{W}(\mathbf{RCR}^t)$ for all $\mathbf{R} \in g$, if, and only if, the function $\check{W}$, defined by $\hat{W}(\mathbf{C}) = \check{W}(\mathbf{C}, \mathbf{M})$, is invariant under Orth, i.e., $\check{W}(\mathbf{C}, \mathbf{M}) = \check{W}(\mathbf{QCQ}^t, \mathbf{QMQ}^t)$ for all $\mathbf{Q} \in$ Orth; that is, if and only if $\check{W}$ is a jointly isotropic function of its arguments.*

Proof of sufficiency: Suppose $\check{W}(\mathbf{C},\mathbf{M})$ is invariant under *Orth*. Then, because $g \subset$ *Orth*, it is also invariant under g. Pick $\mathbf{Q} \in g$. Then,

$$\begin{aligned}\hat{W}(\mathbf{QCQ}') &= \check{W}(\mathbf{QCQ}',\mathbf{M}) = \check{W}(\mathbf{QCQ}',\mathbf{Q}(\mathbf{Q}'\mathbf{MQ})\mathbf{Q}')\\ &= \check{W}(\mathbf{C},\mathbf{Q}'\mathbf{MQ}) \quad \text{(invariance under } g)\\ &= \check{W}(\mathbf{C},\mathbf{M}) \quad (\text{because } \mathbf{Q}' \in g),\end{aligned} \tag{5.7}$$

and thus $\hat{W}(\mathbf{QCQ}') = \hat{W}(\mathbf{C})$, i.e., $\hat{W}$ is invariant under g.

Proof of necessity: Before proceeding, we define a function

$$\tilde{W}(\mathbf{C},\mathbf{P}) = \hat{W}(\mathbf{RCR}'), \tag{5.8}$$

where $\mathbf{R} \in$ *Orth* satisfies $\mathbf{RPR}' = \mathbf{M}$, i.e., $\mathbf{P} = \mathbf{R}'\mathbf{MR}$. Note that if $\mathbf{R} \in g$, then $\mathbf{P} = \mathbf{M}$ and $\tilde{W}(\mathbf{C},\mathbf{P})$ reduces to $\hat{W}(\mathbf{C}) = \check{W}(\mathbf{C},\mathbf{M})$. Thus, $\tilde{W}$ defines an *extension* of $\check{W}$ from g to *Orth*. We now show that $\tilde{W}$ is invariant under *Orth*.

For any $\mathbf{Q} \in$ *Orth*, by the definition of $\tilde{W}$,

$$\tilde{W}(\mathbf{QCQ}',\mathbf{QPQ}') = \hat{W}(\mathbf{R}(\mathbf{QCQ}')\mathbf{R}'), \tag{5.9}$$

where $\mathbf{R} \in$ *Orth* satisfies $\mathbf{RQP}(\mathbf{RQ})' = \mathbf{M}$. Let $\bar{\mathbf{R}} = \mathbf{RQ}$. Then $\bar{\mathbf{R}}\mathbf{P}\bar{\mathbf{R}}' = \mathbf{M}$, and the definition of $\tilde{W}$ yields

$$\tilde{W}(\mathbf{C},\mathbf{P}) = \hat{W}(\bar{\mathbf{R}}\mathbf{C}\bar{\mathbf{R}}'). \tag{5.10}$$

Thus, $\tilde{W}(\mathbf{QCQ}',\mathbf{QPQ}') = \tilde{W}(\mathbf{C},\mathbf{P})$, and so $\tilde{W}$ is invariant under *Orth*.

To summarize, we have shown that $\hat{W}(\mathbf{C})$ is invariant under $g \Leftrightarrow \tilde{W}$ is an isotropic function. We turn now to our main result, the:

Representation theorem for transverse isotropy: $\hat{W}(\mathbf{C})$ is a function of $I_k(\mathbf{C})$; $k = 1,2,3$, and of

$$I_4(\mathbf{C}) = \mathbf{m}\cdot\mathbf{Cm} = \mathbf{C}\cdot\mathbf{M} \quad \text{and} \quad I_5(\mathbf{C}) = \mathbf{m}\cdot\mathbf{C}^2\mathbf{m} = \mathbf{C}^2\cdot\mathbf{M}. \tag{5.11}$$

Proof of sufficiency: We know that $I_k(\mathbf{QCQ}') = I_k(\mathbf{C})$; $k = 1,2,3$, for all $\mathbf{Q} \in$ *Orth*. Furthermore,

$$tr[\mathbf{QCQ}'(\mathbf{QMQ}')] = tr(\mathbf{QCMQ}') = tr(\mathbf{Q}'\mathbf{QCM}) = tr(\mathbf{CM}), \tag{5.12}$$

and the same is true if $\mathbf{C}$ is replaced by $\mathbf{C}^2$; accordingly, $I_k(\mathbf{QCQ}',\mathbf{QMQ}') = I_k(\mathbf{C},\mathbf{M})$; $k = 4,5$. It follows that if

$$\check{W}(\mathbf{C},\mathbf{M}) = U(I_1,...,I_5), \tag{5.13}$$

then $\check{W}(\mathbf{C},\mathbf{M}) = \check{W}(\mathbf{QCQ}',\mathbf{QMQ}')$.

Proof of necessity: Suppose $\check{W}(\mathbf{C},\mathbf{M}) = \check{W}(\mathbf{QCQ}^t, \mathbf{QMQ}^t)$ for all $\mathbf{Q} \in Orth$. We want to show that $\check{W}(\bar{\mathbf{C}},\bar{\mathbf{M}}) = \check{W}(\mathbf{C},\mathbf{M})$ whenever $I_k(\bar{\mathbf{C}},\bar{\mathbf{M}}) = I_k(\mathbf{C},\mathbf{M})$; $k = 1, ..., 5$. For, $\check{W}(\mathbf{C},\mathbf{M})$ is then determined by the list $I_k(\mathbf{C},\mathbf{M})$; $k = 1, ..., 5$.

Proceeding, consider any symmetric tensors $\mathbf{A}$ and $\bar{\mathbf{A}}$, and any unit vectors $\mathbf{n}$ and $\bar{\mathbf{n}}$. Let $\mathbf{N} = \mathbf{n} \otimes \mathbf{n}$ and $\bar{\mathbf{N}} = \bar{\mathbf{n}} \otimes \bar{\mathbf{n}}$, and suppose

$$\begin{aligned} I_k(\mathbf{A}) &= I_k(\bar{\mathbf{A}}); \quad k = 1,2,3, \\ \mathbf{n}\cdot\mathbf{n} &= \bar{\mathbf{n}}\cdot\bar{\mathbf{n}} \quad (tr\mathbf{N} = tr\bar{\mathbf{N}}), \\ \mathbf{n}\cdot\mathbf{A}\mathbf{n} &= \bar{\mathbf{n}}\cdot\bar{\mathbf{A}}\bar{\mathbf{n}} \quad (tr(\mathbf{AN}) = tr(\bar{\mathbf{A}}\bar{\mathbf{N}})), \\ \mathbf{n}\cdot\mathbf{A}^2\mathbf{n} &= \bar{\mathbf{n}}\cdot\bar{\mathbf{A}}^2\bar{\mathbf{n}} \quad (tr(\mathbf{A}^2\mathbf{N}) = tr(\bar{\mathbf{A}}^2\bar{\mathbf{N}})). \end{aligned} \tag{5.14}$$

From the first of these we conclude that $\mathbf{A}$ and $\bar{\mathbf{A}}$ have the same eigenvalues; therefore,

$$\mathbf{A} = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i \quad \text{and} \quad \bar{\mathbf{A}} = \sum_{i=1}^{3} \lambda_i \bar{\mathbf{u}}_i \otimes \bar{\mathbf{u}}_i, \tag{5.15}$$

where $\{\mathbf{u}_i\}$ and $\{\bar{\mathbf{u}}_i\}$ are orthonormal triads. Thus,

$$\mathbf{Q} \equiv \bar{\mathbf{u}}_i \otimes \mathbf{u}_i \in Orth, \tag{5.16}$$

and

$$\bar{\mathbf{A}} = \mathbf{QAQ}^t. \tag{5.17}$$

From eqn $(5.14)_{2,3,4}$ it follows that for any scalars α, β, γ,

$$\bar{\mathbf{n}}\cdot(\alpha\bar{\mathbf{I}} + \beta\bar{\mathbf{A}} + \gamma\bar{\mathbf{A}}^2)\bar{\mathbf{n}} = \mathbf{n}\cdot(\alpha\mathbf{I} + \beta\mathbf{A} + \gamma\mathbf{A}^2)\mathbf{n}, \tag{5.18}$$

where $\bar{\mathbf{I}} = \mathbf{QIQ}^t = \mathbf{I}$, i.e.,

$$\mathbf{n}\cdot(\alpha\mathbf{I} + \beta\mathbf{A} + \gamma\mathbf{A}^2)\mathbf{n} = \mathbf{Q}^t\bar{\mathbf{n}}\cdot(\alpha\mathbf{I} + \beta\mathbf{A} + \gamma\mathbf{A}^2)\mathbf{Q}^t\bar{\mathbf{n}}. \tag{5.19}$$

We will prove the theorem under the restriction that the eigenvalues are distinct, leaving the general case to the interested reader. Before proceeding, we pause to verify a:

Lemma: $\mathbf{I}$, $\mathbf{A}$ and $\mathbf{A}^2$ are linearly independent, and

$$\mathcal{S} = Span\{\mathbf{I},\ \mathbf{A},\ \mathbf{A}^2\}, \quad \text{where} \quad \mathcal{S} \equiv Span\{\mathbf{u}_1 \otimes \mathbf{u}_2, \mathbf{u}_2 \otimes \mathbf{u}_2, \mathbf{u}_3 \otimes \mathbf{u}_3\}. \tag{5.20}$$

The proof is standard and may be found, for example, in Gurtin (1981). We sketch it here. Assume that

$$a_0\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2 = \mathbf{0}, \tag{5.21}$$

with $\{a_0, a_1, a_2\} \neq \{0, 0, 0\}$. This is equivalent to

$$\sum_{i=1}^{3}(a_0 + a_1\lambda_i + a_2\lambda_i^2)\mathbf{u}_i \otimes \mathbf{u}_i = \mathbf{0}, \tag{5.22}$$

and hence to

$$a_0 + a_1\lambda_i + a_2\lambda_i^2 = 0; \quad i = 1, 2, 3. \tag{5.23}$$

This means that the three distinct λ_i all satisfy the same quadratic equation, which has at most two distinct roots. This contradiction leads to the conclusion that eqn (5.21) is true if and only if $\{a_0, a_1, a_2\} = \{0, 0, 0\}$, and hence that the set $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2\}$ is linearly independent. Furthermore, by the spectral representation (5.15), first part, we have $\mathbf{I}, \mathbf{A}, \mathbf{A}^2 \in \mathcal{S}$, a three-dimensional vector space. As any set of of n linearly-independent elements of an n-dimensional vector space constitutes a basis for that vector space, eqn (5.20) follows, and we conclude that every $\mathbf{B} \in \mathcal{S}$ satisfies

$$\mathbf{B} = \alpha\mathbf{I} + \beta\mathbf{A} + \gamma\mathbf{A}^2 \tag{5.24}$$

for some (unique) α, β, γ.

Returning to the theorem, we see that eqn (5.19) is equivalent to the statement

$$\mathbf{n} \cdot \mathbf{B}\mathbf{n} = \mathbf{Q}^t\bar{\mathbf{n}} \cdot \mathbf{B}(\mathbf{Q}^t\bar{\mathbf{n}}) \tag{5.25}$$

for all $\mathbf{B} \in \mathcal{S}$. Let $\mathbf{R} \in Orth$ be such that $\bar{\mathbf{n}} = \mathbf{R}\mathbf{n}$. Then, $\bar{\mathbf{N}} = \mathbf{R}\mathbf{N}\mathbf{R}^t$ and eqn (5.25) reduces to

$$\mathbf{D} \cdot \mathbf{N} = 0 \quad \text{for all} \quad \mathbf{N} = \mathbf{n} \otimes \mathbf{n}, \tag{5.26}$$

where

$$\mathbf{D} = \mathbf{B} - (\mathbf{Q}^t\mathbf{R})^t\mathbf{B}(\mathbf{Q}^t\mathbf{R}) \in Sym. \tag{5.27}$$

Let $\mathbf{n}_i$ be the eigenvectors of an arbitrary symmetric tensor $\mathbf{S}$, i.e., $\mathbf{S} = \sum_{i=1}^{3} S_i\mathbf{N}_i$, where S_i are the corresponding eigenvalues and $\mathbf{N}_i = \mathbf{n}_i \otimes \mathbf{n}_i$. From eqn (5.26) we conclude that $\mathbf{D} \in Sym$ satisfies $\mathbf{D} \cdot \mathbf{S} = 0$ for all $\mathbf{S} \in Sym$ and, hence, that $\mathbf{D}$ vanishes, i.e., that

$$(\mathbf{Q}^t\mathbf{R})^t\mathbf{B}(\mathbf{Q}^t\mathbf{R}) = \mathbf{B}, \quad \text{for all} \quad \mathbf{B} \in \mathcal{S}. \tag{5.28}$$

Now, every $\mathbf{B} \in \mathcal{S}$ is expressible in the form $\mathbf{B} = \sum_{i=1}^{3} B_i\mathbf{u}_i \otimes \mathbf{u}_i$ for some scalars B_i. Then, since eqn (5.28) holds for any $\mathbf{B} \in \mathcal{S}$, it is necessary and sufficient that

$$\mathbf{R}^t\mathbf{Q}(\mathbf{u} \otimes \mathbf{u}) = (\mathbf{u} \otimes \mathbf{u})\mathbf{R}^t\mathbf{Q}, \tag{5.29}$$

for all $\mathbf{u} \otimes \mathbf{u} \in \{\mathbf{u}_1 \otimes \mathbf{u}_2, \mathbf{u}_2 \otimes \mathbf{u}_2, \mathbf{u}_3 \otimes \mathbf{u}_3\}$. Thus, $(\mathbf{R}^t\mathbf{Q}\mathbf{u}) \otimes \mathbf{u} = \mathbf{u} \otimes (\mathbf{Q}^t\mathbf{R}\mathbf{u})$, which yields

$$\mathbf{R}^t\mathbf{Q}\mathbf{u} = (\mathbf{u} \cdot \mathbf{R}^t\mathbf{Q}\mathbf{u})\mathbf{u} = (\mathbf{R}^t\mathbf{Q}\mathbf{u} \cdot \mathbf{u})\mathbf{u}. \tag{5.30}$$

However, $|\mathbf{R}^t\mathbf{Q}\mathbf{u}| = |\mathbf{u}|$ because $\mathbf{R}^t\mathbf{Q} \in Orth$. Thus, $|\mathbf{u}|^2 = \mathbf{R}^t\mathbf{Q}\mathbf{u} \cdot \mathbf{R}^t\mathbf{Q}\mathbf{u} = (\mathbf{R}^t\mathbf{Q}\mathbf{u} \cdot \mathbf{u})^2 |\mathbf{u}|^2$, so that $\mathbf{u} \cdot \mathbf{R}^t\mathbf{Q}\mathbf{u} = \pm 1$. Then, $\mathbf{R}^t\mathbf{Q}\mathbf{u}_i = \pm\mathbf{u}_i$, or $\mathbf{Q}\mathbf{u}_i = \pm\mathbf{R}\mathbf{u}_i$; $i = 1, 2, 3$. Finally,

$$\mathbf{R}\mathbf{A}\mathbf{R}^t = \sum_{i=1}^{3} \lambda_i \mathbf{R}\mathbf{u}_i \otimes \mathbf{R}\mathbf{u}_i = \sum_{i=1}^{3} \lambda_i \mathbf{Q}\mathbf{u}_i \otimes \mathbf{Q}\mathbf{u}_i = \mathbf{Q}\mathbf{A}\mathbf{Q}^t = \bar{\mathbf{A}}. \tag{5.31}$$

To summarize, we have shown that if eqn (5.14), parts 1–4, hold, then

$$\bar{\mathbf{N}} = \mathbf{R}\mathbf{N}\mathbf{R}^t \quad \text{and} \quad \bar{\mathbf{A}} = \mathbf{R}\mathbf{A}\mathbf{R}^t \tag{5.32}$$

for some $\mathbf{R} \in Orth$. Then,

$$\check{W}(\mathbf{A}, \mathbf{N}) = \check{W}(\mathbf{R}\mathbf{A}\mathbf{R}^t, \mathbf{R}\mathbf{N}\mathbf{R}^t) = \check{W}(\bar{\mathbf{A}}, \bar{\mathbf{N}}), \tag{5.33}$$

and we conclude that $\check{W}(\mathbf{A}, \mathbf{N})$ is determined by $I_k(\mathbf{A})$; $k = 1, 2, 3$, and by $\mathbf{n} \cdot \mathbf{A}\mathbf{n}$, $\mathbf{n} \cdot \mathbf{A}^2\mathbf{n}$ and $\mathbf{n} \cdot \mathbf{n}$, the last of these being redundant if $\mathbf{n}$ is a unit vector.

Our outline of this representation theorem follows the proof given in the paper by Liu (1982). The papers by Boehler (1979) and Zheng (1994) should also be consulted. The method of the theorem may be extended to cover any type of symmetry that can be characterized by a set of structural tensors, i.e., by tensors $\mathbf{S}$ such that $\mathbf{R}\mathbf{S}\mathbf{R}^t = \mathbf{S}$ for all $\mathbf{R} \in g$. In fact, the latter restriction may be relaxed, as shown in the paper by Man and Goddard (2017). The general issue of material symmetry and attendant representation theorems is discussed in a series of fundamental papers by Rivlin and associates (Barenblatt and Joseph (1997)).

To use the present representation theorem in the context of the theory of elasticity for transversely isotropic materials, we impose

$$\hat{W}(\mathbf{C}) = U(I_1, ..., I_5), \tag{5.34}$$

and find, for any parametrized path of deformations, that

$$Sym\hat{W}_{\mathbf{C}} \cdot \dot{\mathbf{C}} = (\hat{W})^{\cdot} = \sum_{k=1}^{5} U_k \dot{I}_k = \sum_{k=1}^{5} U_k Sym(I_k)_{\mathbf{C}} \cdot \dot{\mathbf{C}}, \tag{5.35}$$

with $U_k = \partial U / \partial I_k$. Here, we have

$$\dot{I}_4 = \mathbf{M} \cdot \dot{\mathbf{C}} \tag{5.36}$$

and

$$\dot{I}_5 = \mathbf{M} \cdot (\mathbf{C}^2)^{\cdot} = (\mathbf{C}\dot{\mathbf{C}} + \dot{\mathbf{C}}\mathbf{C}) \cdot \mathbf{M} = (\mathbf{M}\mathbf{C} + \mathbf{C}\mathbf{M}) \cdot \dot{\mathbf{C}}, \tag{5.37}$$

and, therefore,

$$Sym(I_4)_C = \mathbf{M} \quad \text{and} \quad Sym(I_5)_C = \mathbf{MC} + \mathbf{CM}. \tag{5.38}$$

Combining these results with eqns (3.11) and (3.20), we derive the constitutive representations

$$\mathbf{P} = 2\mathbf{F}[(U_1 + I_1 U_2)\mathbf{I} - U_2\mathbf{C} + U_3\mathbf{C}^* + U_4\mathbf{m}\otimes\mathbf{m} + U_5(\mathbf{Cm}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{Cm})] \tag{5.39}$$

and

$$J\mathbf{T} = 2[(U_1 + I_1 U_2)\mathbf{B} - U_2\mathbf{B}^2 + I_3 U_3\mathbf{I} + U_4\mathbf{Fm}\otimes\mathbf{Fm} + U_5(\mathbf{BFm}\otimes\mathbf{Fm} + \mathbf{Fm}\otimes\mathbf{BFm})], \tag{5.40}$$

the second of which may be compared to eqn (4.39).

The paper by Horgan and Murphy (2016) describes an interesting application of this model to a specific boundary-value problem, which highlights the unusual features of transverse isotropy. Transverse isotropy plays a large role in the study of bioelasticity. The collection edited by Dorfmann and Ogden (2015) provides a thorough account.

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6

Stress response in the presence of local constraints on the deformation

Quite often the conditions of the problem at hand and the nature of the material are such that the deformation conforms very nearly to one or more constraints on its gradient. Thus, for example, rubber-like solids are nearly incompressible and so deform isochorically provided that, in doing so, no boundary data are violated. From the phenomenological point of view, such behavior is due to the significant energetic cost associated with deviations from a locally isochoric mode of deformation. Indeed, this cost is often so high that if position is assigned on the boundary in such a way as to require an overall volume change, the material will rupture locally, rather than maintain a smooth, and necessarily non-isochoric, deformation. Roughly, isochoric deformations are energetically optimal in rubber-like solids. One may imagine that the application of an arbitrary pressure to such a solid would not affect its deformation to any significant degree, and not at all in the limit of perfect incompressibility. Conversely, the pressure acting on the material is not determined by its deformation. In the same way, deformations of directionally reinforced solids, such as fiber composites, may be idealized as being inextensible along the local direction of reinforcement, the uniaxial stress along this direction being unrelated constitutively to deformations of the material that are consistent with the constraint. These are examples of useful constraints in which the deformation gradient is restricted *a priori*. Because they narrow the class of admissible deformations, they invariably aid in the analytical treatment of problems. This is illustrated in Chapter 7.

For example, if the material is incompressible during a time interval $\mathcal{I}$ then we have $J(t) = const.$ for $t \in \mathcal{I}$. Differentiating and using $J_{\mathbf{F}} = \mathbf{F}^*$ yields $\mathbf{F}^* \cdot \dot{\mathbf{F}} = 0$. This imposes a restriction on $\dot{\mathbf{F}}$, and so the argument leading from eqn (3.10) to (3.11) no longer holds. Evidently, the manner in which the stress is related to the deformation is thus modified. Our purpose, here, is to determine how it is modified. The subject is not especially well treated in the text and monograph literatures, and so we present a systematic discussion of it here, based on the Lagrange-multiplier theorem.

Finite Elasticity Theory. David J. Steigmann.

6.1 Local constraints

We consider local constraints at material point p, as perceived by observer $\mathcal{O}$, of the form

$$\phi_\kappa(\mathbf{F}) = 0. \tag{6.1}$$

It is natural to assume that all observers, $\mathcal{O}^+$ included, agree that a constraint is in force and thus to require that an expression of the kind

$$\phi^+_{\kappa^+}(\mathbf{F}^+) = 0 \tag{6.2}$$

hold whenever eqn (6.1) does. Because constraints reflect the nature of the material under certain conditions, on which $\mathcal{O}$ and $\mathcal{O}^+$ are presumed to agree, we may follow the example of the strain–energy function in Problem 1 of Chapter 3 and assert that $\phi^+_{\kappa^+}(\mathbf{F}^+) = \phi_\kappa(\mathbf{F})$. If you have worked through that simple exercise then you know that this implies $\phi_\kappa(\mathbf{F}) = \phi_\kappa(\mathbf{QF})$ for all *rotations* $\mathbf{Q}$ and that the latter is equivalent to

$$\phi_\kappa(\mathbf{F}) = \psi_\kappa(\mathbf{C}), \tag{6.3}$$

for some function ψ_κ. The symmetry of $\mathbf{C}$ implies that there can be no more than six independent constraints at any material point. For, otherwise the constraints would overspecify the components of $\mathbf{C}$.

Problem

Show that there can be no non-trivial constraints of the form $\mathbf{A} \cdot \mathbf{F} = B$ with $\mathbf{A}$ and B fixed.

For example, incompressibility requires that the value of J at a material point be the same in all configurations of the body. This is equivalent to the requirement that $\det \mathbf{C}$ be independent of the deformation. Then, since $\mathbf{C} = \mathbf{I}$, when the body is undeformed, the constraint function is given by

$$\psi_\kappa(\mathbf{C}) = \det \mathbf{C} - 1. \tag{6.4}$$

In the case of inextensibility, $|\mathbf{FE}|$ is unaffected by deformation, where $\mathbf{E}$ is the unit-tangent vector to an inextensible material curve in κ. Its value is unity in that configuration, and the constraint function is thus given by

$$\psi_\kappa(\mathbf{C}) = \sqrt{\mathbf{E} \otimes \mathbf{E} \cdot \mathbf{C}} - 1. \tag{6.5}$$

6.2 Constraint manifolds and the Lagrange multiplier rule

Evidently eqn (6.1) defines a manifold $\mathcal{M}$ in the nine-dimensional space Lin^+, just as an equation of the form $F(\mathbf{x}) = 0$ defines a surface in three-space. If there are n constraints

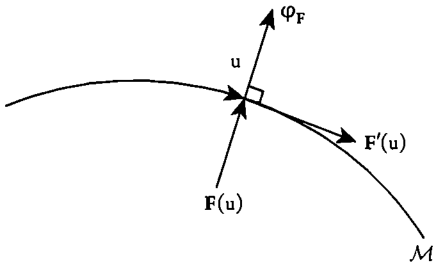

Figure 6.1 Local geometry of the constraint manifold

$\phi_\kappa^{(i)}(\mathbf{F})$ then $\mathbf{F} \in \mathcal{M}$, where $\mathcal{M} = \cap_{i=1}^{n} \mathcal{M}_i$ and $\mathcal{M}_i = \{\mathbf{F} : \phi_\kappa^{(i)}(\mathbf{F}) = 0\}$. $\mathcal{M}$ is called the *constraint manifold*. Since $J(= \det \mathbf{F})$ is non-zero in any deformation we require that $\mathbf{0} \notin \mathcal{M}$. Therefore, $\mathcal{M}$ is not a linear space.

On any curve $\mathbf{F}(u) \in \mathcal{M}$, the stress and strain energy are related by

$$(W(\mathbf{F}))' = \mathbf{P} \cdot \mathbf{F}', \tag{6.6}$$

as in eqn (3.9), where $\mathbf{F}' \in T_{\mathcal{M}}$, the vector space tangent to the constraint manifold at the point $\mathbf{F}(u)$. We assume that each point $\mathbf{F}$ on $\mathcal{M}$ is the center of an open ball $\mathcal{B}$ in Lin^+. Further, for any possible process we have $\phi_\kappa^{(i)\prime}(u) = 0$ and therefore $\phi_{\mathbf{F}}^{(i)} \cdot \mathbf{F}' = 0$ for all $\mathbf{F}' \in T_{\mathcal{M}}$, where the gradients are evaluated at the point $\mathbf{F}(u)$ and we have suppressed the subscript κ for clarity. This implies that each of the gradients $\phi_{\mathbf{F}}^{(i)}$ is orthogonal to $T_{\mathcal{M}}$ (see Figure 6.1).

By definition, the constraints are independent if and only if the set $\{\phi_{\mathbf{F}}^{(i)}\}$ is linearly independent; that is, if and only if the linear equation

$$\sum_{i=1}^{n} \alpha_i \phi_{\mathbf{F}}^{(i)} = \mathbf{0} \tag{6.7}$$

holds with all $\alpha_i = 0$. In this case, $\{\phi_{\mathbf{F}}^{(i)}\}$ is a basis for the *orthogonal complement* to $T_{\mathcal{M}}$. The tangent space and its orthogonal complement together comprise the nine-dimensional *translation space* of Lin^+, the linear (vector) space consisting of all differences that can be formed from the elements of Lin^+. We have already seen that this is just Lin, and so

$$Lin = T_{\mathcal{M}} \oplus Span\{\phi_{\mathbf{F}}^{(i)}\}. \tag{6.8}$$

The notation $\oplus$ identifies Lin as the *direct sum* of the vector subspaces $T_{\mathcal{M}}$ and $Span\{\phi_{\mathbf{F}}^{(i)}\}$, meaning that every element of Lin is expressible as the sum of elements of the two vector spaces comprising the direct sum. Of course, direct-sum decompositions are not unique. Two that come immediately to mind in the case of Lin are $Sym \oplus Skw$, the direct sum of the linear spaces of symmetric and skew tensors, and *Sph* ⊕ *Dev*, in which the factors are, respectively, the linear spaces consisting of the spherical and deviatoric tensors.

Thus, any $\mathbf{A} \in Lin$ such that $\mathbf{A} \cdot \mathbf{F}' = 0$ for all $\mathbf{F}' \in T_{\mathcal{M}}$ satisfies

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \phi_{\mathbf{F}}^{(i)} \tag{6.9}$$

for some scalars λ_i. Equation (6.6) and the chain rule imply that $\mathbf{P} - W_{\mathbf{F}}$ furnishes an example of such tensors, and hence that

$$\mathbf{P} = W_{\mathbf{F}} + \sum_{i=1}^{n} \lambda_i \phi_{\mathbf{F}}^{(i)}, \tag{6.10}$$

where the *Lagrange multipliers* λ_i may depend on the material point p and the time t, which have played passive roles in course of the derivation. Because the domain of W is $\mathcal{M}$, the derivative $W_{\mathbf{F}}$, defined by $W' = W_{\mathbf{F}} \cdot \mathbf{F}'$, is to be interpreted as an element of the dual space of $T_{\mathcal{M}}$, which may be assumed to coincide with $T_{\mathcal{M}}$ itself. We do this all the time, for example, as when we ignore the distinction between three-dimensional Euclidean space and its dual. In contrast, the functions $\phi^{(i)}(\mathbf{F})$ are defined on Lin^+ and their gradients belong to *Lin*.

Typically, one wishes to compute the gradient $W_{\mathbf{F}}$ explicitly via the chain rule, as in Part 3 of the Supplement. When doing this, the fact that the associated $\mathbf{F}'$ is not an arbitrary element of *Lin* may prove to be an inconvenience. We may effectively sidestep this issue by using a smooth *extension* $\bar{W}$ of W instead. The extended function has the ball $\mathcal{B}$ as its domain, is differentiable there, and by definition, agrees with W on $\mathcal{M}$. Differentiating the consequent equation yields $(\bar{W})' = W'$ at all points $\mathbf{F}(u) \in \mathcal{M}$, so that

$$(W_{\mathbf{F}} - \bar{W}_{\mathbf{F}}) \cdot \mathbf{F}' = 0, \tag{6.11}$$

and therefore $W_{\mathbf{F}} - \bar{W}_{\mathbf{F}} \in Span\{\phi_{\mathbf{F}}^{(i)}\}$. The use of $\bar{W}$ in place of W in the formula (6.10) for the stress thus amounts to an adjustment to the (as yet unknown) Lagrange multipliers. Moreover, if $\tilde{W}$ is another extension, then, because it agrees with $\bar{W}$ on $\mathcal{M}$, it follows that eqn (6.11) remains valid with $\tilde{W}$ substituted in place of $\bar{W}$, and eqn (6.10) continues to hold with possibly different multipliers. Therefore, *any* smooth extension may be used without loss of generality. The obvious choice, and the one tacitly made in all treatments of constrained elasticity, is

$$\bar{W}(\mathbf{F}) = W(\mathbf{F}), \quad \text{for} \quad \mathbf{F} \in Lin^+. \tag{6.12}$$

That is, the extended function may be taken to be the original function, but now with domain Lin^+ rather than $\mathcal{M}$.

Since W is a constitutive function, it is subject to invariance requirements. Proceeding as before, we conclude that

$$W(\mathbf{F}) = G(\mathbf{C}), \quad \text{and therefore} \quad \bar{W}(\mathbf{F}) = \bar{G}(\mathbf{C}) \tag{6.13}$$

for some G defined on the image of $\mathcal{M}$ in Sym^+, with

$$\bar{G}(\mathbf{C}) = G(\mathbf{C}) \quad \text{for } all \quad \mathbf{C} \in Sym^+, \tag{6.14}$$

in accordance with eqn (6.12). We compute $\mathbf{P} = \mathbf{FS}$ where

$$\frac{1}{2}\mathbf{S} = SymG_{\mathbf{C}} + \sum_{i=1}^{n} \lambda_i Sym\psi_{\mathbf{C}}^{(i)} \tag{6.15}$$

in which $\psi^{(i)}(\mathbf{C}) = \phi^{(i)}(\mathbf{F})$. Here, we have used the formula eqn (5.4) of the Supplement. This is justified because the extended strain-energy function is defined for $\mathbf{F} \in Lin^+$ and the induced $\mathbf{F}'$ is an arbitrary element of Lin. Accordingly,

$$\mathbf{P} = 2\mathbf{F}\left[SymG_{\mathbf{C}} + \sum_{i=1}^{n} \lambda_i Sym\psi_{\mathbf{C}}^{(i)}\right]. \tag{6.16}$$

In the example of incompressibility we find, from eqn (6.4), that

$$Sym\psi_{\mathbf{C}} = (\det \mathbf{C})\mathbf{C}^{-1} \tag{6.17}$$

and for inextensibility we use $|\mathbf{FE}| = \sqrt{\mathbf{E} \otimes \mathbf{E} \cdot \mathbf{C}}$ with the chain rule to derive

$$Sym\psi_{\mathbf{C}} = \frac{1}{2}|\mathbf{FE}|^{-1}\,\mathbf{E} \otimes \mathbf{E}. \tag{6.18}$$

The gradients of the associated functions of $\mathbf{F}$ are obtained by using the formula

$$\phi_{\mathbf{F}} = 2\mathbf{F}(Sym\psi_{\mathbf{C}}), \tag{6.19}$$

which is derived just as eqn (3.20) was derived. Then, in the case of incompressibility,

$$\phi_{\mathbf{F}} = 2(\det \mathbf{C})\mathbf{F}^{-t}, \tag{6.20}$$

whereas, for inextensibility,

$$\phi_{\mathbf{F}} = |\mathbf{FE}|^{-1}\,\mathbf{FE} \otimes \mathbf{E}. \tag{6.21}$$

It is easy to verify that these are linearly independent elements of Lin and, thus, that the two constraints are independent.

When using eqn (6.10) to compute the stress, all gradients are evaluated at $\mathbf{F} \in \mathcal{M}$. Accordingly, for incompressibility and inextensibility we have

$$\mathbf{P} = W_{\mathbf{F}} - p\mathbf{F}^{-t} \quad \text{and} \quad \mathbf{P} = W_{\mathbf{F}} + \lambda \mathbf{e} \otimes \mathbf{E}, \tag{6.22}$$

respectively, where p and λ are the Lagrange multipliers and $\mathbf{e} = \mathbf{FE}$ is the unit tangent to the inextensible curve after deformation. The associated Cauchy stresses are

$$\mathbf{T} = (W_{\mathbf{F}})\mathbf{F}^t - p\mathbf{I} \quad \text{and} \quad \mathbf{T} = J^{-1}(W_{\mathbf{F}})\mathbf{F}^t + T\mathbf{e} \otimes \mathbf{e}, \tag{6.23}$$

where $T = J^{-1}\lambda$. This yields the interpretation of the Lagrange multipliers as a pure pressure in the first instance, and a uniaxial stress in the second. Recalling our earlier discussion, the fact that these are unrelated to the deformation is only to be expected. If both constraints are operative, then of course, the stress is obtained by simply adding the constraint terms in accordance with eqn (6.10).

To evaluate the Lagrange multipliers, which at this stage are arbitrary scalar functions of $\mathbf{x}$ and t, we append the n constraint equations to the system consisting of the equations of motion and the boundary and initial conditions. This yields a formally determinate problem consisting of $3 + n$ equations for the three components of the deformation function $\chi(\mathbf{x}, t)$ and the n Lagrange multipliers. In this way the multipliers are found to be influenced by material constitution only indirectly via the initial-boundary-value problem at hand.

6.3 Material symmetry in the presence of constraints

Recall that $\mathbf{R} \in g_{\kappa(p)}$, if and only if, $W_\kappa(\mathbf{F}; \mathbf{x}) = W_\kappa(\mathbf{FR}; \mathbf{x})$. For constrained materials, this statement makes sense only if $\mathbf{F} \in \mathcal{M}$ implies that $\mathbf{FR} \in \mathcal{M}$. Then,

$$\mathbf{R} \in g_{\mathcal{M}} = \{\mathbf{R}: \quad \mathbf{FR} \in \mathcal{M} \text{ whenever } \mathbf{F} \in \mathcal{M}\}, \tag{6.24}$$

and as the statement $\mathbf{R} \in g_{\kappa(p)}$ makes sense only if $\mathbf{R} \in g_{\mathcal{M}}$, it follows that

$$g_{\kappa(p)} \subset g_{\mathcal{M}}. \tag{6.25}$$

Following Podio–Guidugli (2000), we say that the material symmetry is *compatible* with the constraint.

For example, in the case of inextensibility we have $\mathcal{M} = \{\mathbf{F}: \quad |\mathbf{FE}| = 1\}$ in which $\mathbf{E}(\mathbf{x})$ is a field of unit vectors in κ. Then,

$$g_{\mathcal{M}(\textit{inext.})} = \{\mathbf{R}: \quad |(\mathbf{FR})\mathbf{E}| = 1 \text{ whenever } |\mathbf{FE}| = 1\}. \tag{6.26}$$

For incompressibility, $\mathcal{M} = \{\mathbf{F}: \quad \det \mathbf{F} = 1\}$ and

$$g_{\mathcal{M}(\textit{incomp.})} = \{\mathbf{R}: \quad \det(\mathbf{FR}) = 1 \text{ whenever } \det \mathbf{F} = 1\} = U. \tag{6.27}$$

In the case of isotropy, we have $g_\kappa = Orth^+$ and so $\mathbf{R} \in g_{\kappa(iso.)}$ implies $\det(\mathbf{FR}) = \det \mathbf{F}$; thus, $\mathbf{R} \in g_{\mathcal{M}(incomp.)}$ and isotropy is compatible with incompressibility. That is, an isotropic material could be incompressible, although, of course, not every incompressible material is isotropic. In fact, since $g_\kappa \subset U$ is always true, it follows that any kind of symmetry is compatible with incompressibility. On the other hand, for arbitrary $\mathbf{R} \in Orth^+$ we have

$$|(\mathbf{FR})\mathbf{E}| = \sqrt{\mathbf{E} \cdot (\mathbf{R}^t\mathbf{CR})\mathbf{E}} = \sqrt{\mathbf{RE} \cdot \mathbf{C}(\mathbf{RE})} \neq \sqrt{\mathbf{E} \cdot \mathbf{CE}} = |\mathbf{FE}|, \tag{6.28}$$

and so $g_{\kappa(iso.)} \not\subset g_{\mathcal{M}(inext.)}$; isotropy is not compatible with inextensibility and so an isotropic material cannot be inextensible in a fixed direction.

In the case of transverse isotropy we have

$$g_{\kappa(trans.)} = \{\mathbf{R}: \quad \mathbf{R} \in Orth^+ \quad \text{and} \quad \mathbf{RE} = \mathbf{E}\}. \tag{6.29}$$

Then, $|(\mathbf{FR})\mathbf{E}| = |\mathbf{FE}|$ whenever $\mathbf{R} \in g_{\kappa(trans.)}$, implying that $g_{\kappa(trans.)} \subset g_{\mathcal{M}(inext.)}$; transverse isotropy is compatible with inextensibility.

As an example, we cite the case of incompressibility and isotropy. In this case, the natural extension of the strain–energy function is

$$U^*(I_1, I_2; \mathbf{x}) = U(I_1, I_2, 1; \mathbf{x}), \tag{6.30}$$

in which $I_{1,2}$ are the usual invariants of $\mathbf{C}$, defined for *all* $\mathbf{C} \in Sym^+$ in accordance with eqn (6.14). The Cauchy stress is then given by (see eqns (4.39) and (6.23), part 1)

$$\mathbf{T} = 2(U_1^* + I_1 U_2^*)\mathbf{B} - 2U_2^*\mathbf{B}^2 - p\mathbf{I}, \tag{6.31}$$

where p is the Lagrange multiplier. Alternatively, using eqn (4.50) with the extension

$$w^*(i_1, i_2; \mathbf{x}) = w(i_1, i_2, 1; \mathbf{x}), \tag{6.32}$$

in which $i_{1,2}$ are the invariants of $\mathbf{U} \in Sym^+$, we have eqn (4.56) in which

$$\sigma = (w_1^* + i_1 w_2^*)\mathbf{I} - w_2^*\mathbf{U} - p\mathbf{U}^{-1}. \tag{6.33}$$

Problems

1. How is the argument leading from the work inequality to the existence of a strain–energy function affected by a constraint of the form $\phi(\mathbf{F}) = 0$? Are there any restrictions on the extended function $W'(\mathbf{F})$ (the extension of $W(\mathbf{F})$ from $\mathcal{M}$ to Lin^+) arising from the requirement $\mathbf{T} = \mathbf{T}^t$?
2. Find the form of the constitutively indeterminate Cauchy stress for the following local constraints:

(a) A laminated material formed from sheets of stiff paper interspersed in a soft matrix material. Take the sheets to be continuously distributed parallel planes in some reference configuration. Let the planes of the sheets be spanned by an orthonormal set $\{\mathbf{E}_1, \mathbf{E}_2\}$. The constraints are then given by $\mathbf{E}_1 \cdot \mathbf{C}\mathbf{E}_2 = 0$ and $\mathbf{E}_1 \cdot \mathbf{C}\mathbf{E}_1 = 1 = \mathbf{E}_2 \cdot \mathbf{C}\mathbf{E}_2$. Show that these imply there can be no extensional or shear strain in the plane of the sheets, but that transverse normal and shear strains are permitted. Show that a material constrained in this way cannot be isotropic. Could it be transversely isotropic?

(b) The body is laminated as in (a) but the constraint is now that the planes experience no change in local surface area in any deformation of the body. Show that the projection of the Cauchy stress onto the tangent plane of the deformed image of a typical lamina is constitutively indeterminate. Can you describe this in physical terms? What kind of material symmetry is consistent with this constraint?

3. Consider an *incompressible* elastic material that is homogeneous and isotropic relative to the chosen reference configuration. Take this configuration to be a unit cube with edges parallel to $\mathbf{E}_A$, and let the deformation be homogeneous and isochoric with gradient $\mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{E}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{E}_2 + (\lambda_1\lambda_2)^{-1}\mathbf{e}_3 \otimes \mathbf{E}_3$, where λ_1, λ_2 are positive constants and $\{\mathbf{e}_i\} = \{\mathbf{E}_A\}$. Let the tractions vanish on planes with normals $\pm\mathbf{e}_2$ and $\pm\mathbf{e}_3$, and let the forces on planes with normals $\pm\mathbf{e}_1$ be $\pm f\mathbf{e}_1$, respectively.

   (a) Find the constraint pressure in equilibrium in the absence of body force. Show that $\lambda_2 = \lambda_1^{-1/2}$ furnishes a solution, no matter what the strain–energy function may be.

   (b) Obtain f as a function of λ_1 using the so-called neo-Hookean strain-energy function defined by $W = \frac{1}{2}\mu(I_1 - 3)$, where μ is a positive constant (which can be shown to be the shear modulus in the case of small strains). This simple function is quantitatively accurate for rubber if the principal stretches lie in the approximate range $1/2 < \lambda_i < 2$.

4. Generalize the result of Problem 6 in Chapter 4 to the case of incompressibility.

5. How is Problem 7 of Chapter 4 affected by the constraint of incompressibility?

6. Show that all observers agree on the values of the Lagrange multipliers, i.e., that they are absolute scalars.

7. Consider an incompressible elastic solid that is homogeneous and isotropic relative to the chosen reference configuration. Suppose that a material described by this relation is stressed in its *undeformed* state, i.e., $\mathbf{T}$ is non-zero when $\mathbf{F} = \mathbf{I}$. In this case, we say that the material is *residually stressed*. Suppose the residually stressed *undeformed* configuration of the body to be in equilibrium without body force. Also, suppose the traction acting on the *undeformed* body vanishes on a portion of the boundary. Show that the residual stress must then vanish identically everywhere in the body.

REFERENCE

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FURTHER READING

Fleming, W. (1977). *Functions of several variables*. Springer, Berlin.

7

Some boundary-value problems for uniform isotropic incompressible materials

We have already made mention of the fact that analytical solutions to the equations and boundary/initial conditions of nonlinear elasticity theory are as rare as hen's teeth. The youthful student might feel some justification in believing that they are, thus, unworthy of serious study and certainly unworthy, in the digital age, of the often substantial effort required to find them. While it is true that the quest for analytical solutions often requires the investigator to limit attention to rather contrived problems of limited relevance, it is also true that, once secured, they prove to be of the greatest benefit to those seeking to test constitutive equations (for the strain–energy function, say) against empirical data. This is our main justification for considering some simple equilibrium deformations that can be reproduced with relative ease in the laboratory. The best source for analyses of this kind is Ogden (1997), which goes well beyond the present treatment.

7.1 Problems exhibiting radial symmetry with respect to a fixed axis

7.1.1 *Pressurized cylinder*

Take the reference configuration of the elastic material to be the right circular cylinder described in terms of cylindrical polar coordinates by $A \leq R \leq B$ and $0 \leq \theta < 2\pi$. If the cylinder is subjected to uniform pressures at its cylindrical boundaries, and if the material constituting the cylinder is uniform and isotropic, then one has the intuition, based on the cylindrical symmetry of the problem, that cylindrically symmetric deformations should be possible in equilibrium. These are described by a map of the form $\mathbf{y} = \chi(\mathbf{x})$ (dropping the subscript κ for convenience), where

$$\mathbf{x} = R\mathbf{e}_r(\theta) + Z\mathbf{k} \quad \text{and} \quad \mathbf{y} = r\mathbf{e}_r(\theta) + z\mathbf{k}, \tag{7.1}$$

Finite Elasticity Theory. David J. Steigmann.

with

$$\mathbf{e}_r(\theta) = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2 \quad \text{and} \quad \mathbf{k} = \mathbf{e}_3, \tag{7.2}$$

where $\{\mathbf{e}_i\}$ is a fixed orthonormal basis, and

$$r = r(R) \quad \text{and} \quad z = Z. \tag{7.3}$$

This deformation is completely specified by the single function $r(R)$. To visualize it, we observe that it maps a circle $R = C$, say, to the circle $r = c$, where $c = r(C)$ (see Figure 7.1).

To set up the problem of determining $r(R)$, we obtain the deformation gradient and substitute into the relevant constitutive equation. The result is then substituted into the equation of equilibrium and an attempt is made to integrate it. To this end, we use the chain rule $d\mathbf{y} = \mathbf{F}d\mathbf{x}$, where

$$d\mathbf{x} = dR\mathbf{e}_r(\theta) + R\mathbf{e}_\theta(\theta)d\theta + dZ\mathbf{k} \quad \text{and} \quad d\mathbf{y} = dr\mathbf{e}_r(\theta) + r\mathbf{e}_\theta(\theta)d\theta + dz\mathbf{k}, \tag{7.4}$$

where

$$\mathbf{e}_\theta(\theta) = \mathbf{e}_r'(\theta) = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2 = \mathbf{k} \times \mathbf{e}_r(\theta). \tag{7.5}$$

These expressions are entirely general. For the present rather simple class of deformations we use (7.3) to re-write the second of them as

$$d\mathbf{y} = r'(R)dR\mathbf{e}_r(\theta) + r\mathbf{e}_\theta(\theta)d\theta + dZ\mathbf{k}. \tag{7.6}$$

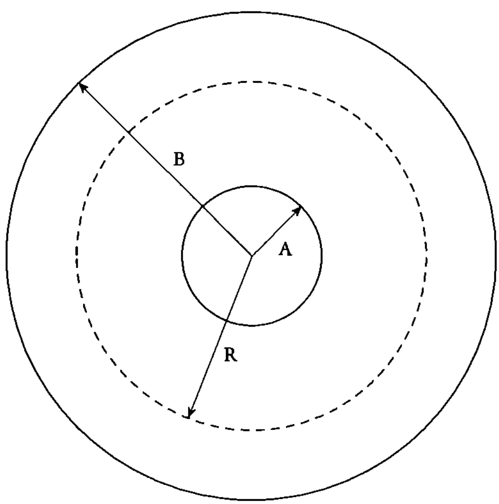

Figure 7.1 Cross section of a cylinder

We want to write this as a tensor operating on the vector $d\mathbf{x}$; that tensor may then be identified with the desired deformation gradient. To achieve this we note, from eqn (7.4) part 1, that

$$dR = \mathbf{e}_r(\theta) \cdot d\mathbf{x}, \quad Rd\theta = \mathbf{e}_\theta(\theta) \cdot d\mathbf{x} \quad \text{and} \quad dZ = \mathbf{k} \cdot d\mathbf{x}, \tag{7.7}$$

and, hence, that

$$d\mathbf{y} = r'(R)\mathbf{e}_r(\theta)[\mathbf{e}_r(\theta) \cdot d\mathbf{x}] + (r/R)\mathbf{e}_\theta(\theta)[\mathbf{e}_\theta(\theta) \cdot d\mathbf{x}] + \mathbf{k}(\mathbf{k} \cdot d\mathbf{x}), \tag{7.8}$$

from which we simply read off

$$\mathbf{F} = r'(R)\mathbf{e}_r(\theta) \otimes \mathbf{e}_r(\theta) + (r/R)\mathbf{e}_\theta(\theta) \otimes \mathbf{e}_\theta(\theta) + \mathbf{k} \otimes \mathbf{k}. \tag{7.9}$$

The restriction given in eqn (1.3) reduces to

$$J = r'(R)(r/R) > 0, \tag{7.10}$$

which implies that $r(R)$ is an increasing function and, hence, that concentric circles $R = C_{1,2}$, with $C_2 > C_1$, are mapped to concentric circles $r = c_{1,2}$, respectively, with $c_2 > c_1$. With this it is trivial to obtain the polar decomposition

$$\mathbf{U} = \mathbf{F}, \quad \mathbf{R} = \mathbf{I}. \tag{7.11}$$

At this stage it is apparent from eqn (7.10) that considerable simplification is achieved if the deformation is isochoric, as it must be if the material is incompressible. Accordingly, we consider incompressibility and integrate eqn (7.10), with $J = 1$, to obtain

$$r^2 - a^2 = R^2 - A^2, \quad \text{where} \quad a = r(A), \tag{7.12}$$

which could have been guessed at the outset. Furthermore, eqn (7.9) furnishes

$$\begin{aligned} \mathbf{F} &= (R/r)\mathbf{e}_r(\theta) \otimes \mathbf{e}_r(\theta) + (r/R)\mathbf{e}_\theta(\theta) \otimes \mathbf{e}_\theta(\theta) + \mathbf{k} \otimes \mathbf{k}, \\ \text{and} \quad \mathbf{B} &= (R/r)^2\mathbf{e}_r(\theta) \otimes \mathbf{e}_r(\theta) + (r/R)^2\mathbf{e}_\theta(\theta) \otimes \mathbf{e}_\theta(\theta) + \mathbf{k} \otimes \mathbf{k}, \end{aligned} \tag{7.13}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^t$ is the left Cauchy–Green deformation tensor.

The representation given in eqn (6.31) for the Cauchy stress in an incompressible, isotropic material leads to

$$\mathbf{T} = \tilde{\mathbf{T}} - p\mathbf{I}, \tag{7.14}$$

where the constitutively determined part of the stress, $\tilde{\mathbf{T}}$, is of the form

$$\tilde{\mathbf{T}} = \tilde{T}_{rr}(r; a)\mathbf{e}_r(\theta) \otimes \mathbf{e}_r(\theta) + \tilde{T}_{\theta\theta}(r; a)\mathbf{e}_\theta(\theta) \otimes \mathbf{e}_\theta(\theta) + \tilde{T}_{zz}(r; a)\mathbf{k} \otimes \mathbf{k}, \tag{7.15}$$

in which $\tilde{T}_{rr}(r; a)$, etc., are obtained by inserting eqn (7.13) into eqn (6.31), while use has been made of eqn (7.12) to convert functions of R into functions of r, depending parametrically on the unknown constant a.

In the absence of body forces, the equation to be solved is $div\mathbf{T} = \mathbf{0}$, which is equivalent to

$$grad p = div\tilde{\mathbf{T}}. \tag{7.16}$$

Problem

Prove the rule $\mathbf{u} \cdot div\mathbf{A} = div(\mathbf{A}^t\mathbf{u}) - \mathbf{A} \cdot grad\mathbf{u}$, where $grad$ is the gradient with respect to position $\mathbf{y}$, and use it to work out the coefficients in the expression $div\mathbf{A} = (\mathbf{e}_r \cdot div\mathbf{A})\mathbf{e}_r + (\mathbf{e}_\theta \cdot div\mathbf{A})\mathbf{e}_\theta + (\mathbf{k} \cdot div\mathbf{A})\mathbf{k}$, where $\mathbf{A} = A_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + A_{r\theta}\mathbf{e}_r \otimes \mathbf{e}_\theta + \ldots$.

Accordingly, we have

$$div\tilde{\mathbf{T}} = \left[\frac{d}{dr}\tilde{T}_{rr} + \frac{1}{r}(\tilde{T}_{rr} - \tilde{T}_{\theta\theta})\right]\mathbf{e}_r(\theta), \tag{7.17}$$

which, in conjunction with

$$grad p = \partial p/\partial r\mathbf{e}_r(\theta) + r^{-1}\partial p/\partial\theta\mathbf{e}_\theta(\theta) + \partial p/\partial z\mathbf{k}, \tag{7.18}$$

leads us to conclude that $\partial p/\partial\theta = 0 = \partial p/\partial z$ and

$$\frac{d}{dr}p(r) = \frac{d}{dr}\tilde{T}_{rr} + \frac{1}{r}(\tilde{T}_{rr} - \tilde{T}_{\theta\theta}) \equiv f(r; a), \tag{7.19}$$

and, therefore, that

$$p(r) = p(a) + \int_a^r f(x; a)dx. \tag{7.20}$$

The boundary conditions at the cylindrical generating surfaces $r = a, b$, with exterior unit normals $\mathbf{n} = \mp\mathbf{e}_r$, respectively, are

$$-\mathbf{T}\mathbf{e}_r = P_a\mathbf{e}_r \quad \text{at} \quad r = a, \quad \text{and} \quad \mathbf{T}\mathbf{e}_r = -P_b\mathbf{e}_r \quad \text{at} \quad r = b, \tag{7.21}$$

where $P_{a,b}$ are the pressures acting there (not to be confused with the boundary values of p). From eqns (7.14) and (7.15), these are seen to be equivalent to the two relations

$$P_a = p(a) - \tilde{T}_{rr}(a; a) \quad \text{and} \quad P_b = p(b) - \tilde{T}_{rr}(b; a), \tag{7.22}$$

where

$$b^2 - a^2 = B^2 - A^2. \tag{7.23}$$

Combining these with eqn (7.20) finally delivers

$$\Delta P = \tilde{T}_{rr}(b; a) - \tilde{T}_{rr}(a; a) - \int_a^b f(r; a)dr, \tag{7.24}$$

where $\Delta P = P_a - P_b$ is the net inflation pressure. For a given strain–energy function, this generates the inflation pressure corresponding to any given radius a.

Problem

Complete the analysis using the so-called neo-Hookean strain–energy function defined by $U = \frac{1}{2}\mu(I_1 - 3)$, where μ, a positive constant, is the shear modulus of the material. This is normalized so as to vanish in the absence of strain, at $I_1 = 3$. Show that the Cauchy stress in a neo-Hookean material is given simply by

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{B}. \tag{7.25}$$

The neo-Hookean model has an interesting history. It actually has a basis in statistical mechanics (see Treloar (1975) and Weiner (2002)), and its relative simplicity makes it attractive to those interested in analytical work. In particular, it is completely specified by the single parameter μ. It is also rather well behaved from the mathematical point of view, as we shall see later in Chapter 9. However, while it furnishes a good quantitative model of rubber for moderate principal stretches lying in the approximate range $(\frac{1}{2}, 2)$, its behavior deviates substantially from that of rubber outside this range. If you have done the preceding exercise about the response of cylinders, you will have observed that it yields a reasonable relationship between inflation pressure and deformed inner radius only when the latter is small-to-moderate. It is, therefore, predictive only for small to moderate strains. In fact, from the empirical point of view, it is no better than the purely phenomenological Varga strain–energy function (see Varga, 1966) defined by

$$w(i_1, i_2) = 2\mu(i_1 - 3), \tag{7.26}$$

with the same μ.

To justify the interpretation of the parameter μ as the shear modulus, we digress to consider the simple-shear deformation defined by

$$\mathbf{y} = \mathbf{x} + \gamma(\mathbf{E}_2 \cdot \mathbf{x})\mathbf{e}_1, \tag{7.27}$$

where γ, the amount of shear, may be any real number, but is assumed here to be independent of $\mathbf{x}$. The effect of this deformation on a unit block of material is illustrated in Figure 7.2.

It is a special case of the deformation analyzed in Problem 7 of Chapter 4.

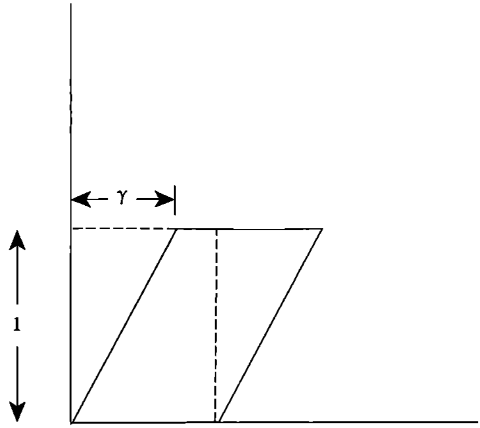

Figure 7.2 Simple shear of a block

The deformation gradient and Cauchy–Green deformation tensor are easily seen to be given by

$$\mathbf{F} = \mathbf{I} + \gamma \mathbf{e}_1 \otimes \mathbf{E}_2 \quad \text{and} \quad \mathbf{C} = \mathbf{I} + \gamma(\mathbf{E}_1 \otimes \mathbf{E}_2 + \mathbf{E}_2 \otimes \mathbf{E}_1) + \gamma^2 \mathbf{E}_2 \otimes \mathbf{E}_2. \tag{7.28}$$

This is an example of a homogeneous deformation. Homogeneous deformations are characterized by the property that the deformation gradient is uniform, i.e., independent of $\mathbf{x}$. Here, the invariants $I_{1,2}$ are determined by the number γ, whereas $J = 1$. In particular, $I_1 = 3 + \gamma^2$, and the neo-Hookean material yields the strain energy $W = \hat{W}(\gamma)$, where

$$\hat{W}(\gamma) = \frac{1}{2}\mu\gamma^2. \tag{7.29}$$

This model thus behaves like a linear spring, with stiffness μ, in simple shear. The energy change associated with a change in the shear is

$$(\hat{W})^{\cdot} = \mathbf{P} \cdot \dot{\mathbf{F}} = (\mathbf{P} \cdot \mathbf{e}_1 \otimes \mathbf{E}_2)\dot{\gamma} = \tau\dot{\gamma}, \tag{7.30}$$

where $\tau = \mathbf{e}_1 \cdot \mathbf{P}\mathbf{E}_2$ is the shear stress, the projection of the (Piola) traction onto the plane with normal $\mathbf{e}_2$. Accordingly, $\tau = \mu\gamma$ and the ratio of shear stress to the amount of shear—the shear modulus—is just μ, as claimed. For a general isotropic material, this ratio depends on γ, in the manner of a nonlinear spring. However, for the neo-Hookean material, the shear response is characterized by a constant modulus and is thus linear; hence, the name neo-*Hookean*.

A strain–energy function that is good over virtually the entire range of feasible deformations of rubber has been given by Ogden (1997). It is rather unwieldy for analytical work and, thus, not discussed in the present chapter, but has emerged as the formulation of choice for numerical simulations. The relevant details can be found in Ogden (1997).

For general isotropic elasticity, including incompressibility, there is a relationship among the stress components in simple shear, which is universal in the sense that it does not involve the properties of the material at hand. This is given simply by specializing the result of Problem 7 of Chapter 4. It predicts that a non-zero normal stress difference always accompanies simple shear. Furthermore, using it one can easily show that the normal stresses vanish faster than the shear stress as the amount of shear vanishes; the normal stress effect is thus inherently nonlinear, which is why one never hears about it in linear elasticity theory. This prediction conforms to empirical observation and is one of the major successes of nonlinear elasticity theory. However, the reader is cautioned that true simple shear is practically unattainable in the laboratory and thus mainly of theoretical interest. Its important features may, however, be replicated in other deformations that are experimentally feasible.

Problem

> Show that the Varga material has a nonlinear simple shear response, and that the parameter μ is the slope of the τ vs γ curve at $\gamma = 0$. Thus, it characterizes the linear part of the shear response of this material at the unstressed state.

Simple shears, and homogeneous deformations in general, are simpler than the cylindrical deformation considered thus far in that, for uniform materials (W is not explicitly dependent on $\mathbf{x}$), they deliver uniform constitutively determined stresses whose divergences vanish identically. This yields $div\mathbf{T} = -gradp$ in the case of incompressibility; therefore, p is uniform if the body is in equilibrium without body forces. The complete stress is then uniform and, thus, determined entirely by boundary data, which must, of course, be such as to admit homogeneous deformations in the interior. Otherwise, the premise is false and the (non-homogeneous) deformation must be found by solving the nonlinear differential equations.

Before leaving deformations of cylinders, we discuss a special case for which the deformation is homogeneous. Thus, consider the case of a solid circular cross section described by $A = 0$ and suppose the deformation is such that $r(0) = 0$. Then eqn (7.12) yields $r = R$ and eqn (7.13) part 1 reduces to $\mathbf{F} = \mathbf{e}_r(\theta) \otimes \mathbf{e}_r(\theta) + \mathbf{e}_\theta(\theta) \otimes \mathbf{e}_\theta(\theta) + \mathbf{k} \otimes \mathbf{k} = \mathbf{I}$. There is no deformation, no matter what the external pressure may be. To make the problem a bit more interesting, we relax the assumption $z = Z$ (cf. eqn (7.3) part 2) and replace it by

$$z = \lambda Z, \tag{7.31}$$

where λ is a constant. An easy calculation yields

$$\mathbf{F} = r'(R)\mathbf{e}_r(\theta) \otimes \mathbf{e}_r(\theta) + (r/R)\mathbf{e}_\theta(\theta) \otimes \mathbf{e}_\theta(\theta) + \lambda\mathbf{k} \otimes \mathbf{k} \tag{7.32}$$

and

$$J = \lambda r'(R)(r/R) > 0 \tag{7.33}$$

in lieu of eqns (7.9) and (7.10). Incompressibility, taken together with the condition on $r(0)$, now furnishes

$$r(R) = R/\sqrt{\lambda}, \tag{7.34}$$

provided that $\lambda > 0$. This, in turn, ensures, as before, that $r(R)$ is an increasing function. Furthermore, eqn (7.32) becomes

$$\mathbf{F} = \lambda \mathbf{k} \otimes \mathbf{k} + \lambda^{-1/2}(\mathbf{I} - \mathbf{k} \otimes \mathbf{k}), \tag{7.35}$$

which is independent of $\mathbf{x}$. The deformation is, therefore, homogeneous and, hence so too, the constitutive part of the stress if the strain–energy function is uniform.

For the neo-Hookean material, the stress is (see eqn (7.25))

$$\mathbf{T} = (\mu\lambda^2 - p)\mathbf{k} \otimes \mathbf{k} + (\mu\lambda^{-1} - p)(\mathbf{I} - \mathbf{k} \otimes \mathbf{k}). \tag{7.36}$$

To find p, which is uniform if the cylinder is in equilibrium with vanishing body force, we need a boundary condition. Suppose, for example, that the lateral surface $R = A(r = a)$ is traction free. Then, $\mathbf{T}\mathbf{e}_r$ vanishes at $r = a$, and hence, in this case, everywhere in the body, yielding $p = \mu\lambda^{-1}$ and

$$\mathbf{T} = \mu(\lambda^2 - \lambda^{-1})\mathbf{k} \otimes \mathbf{k}. \tag{7.37}$$

The stress in the bar is uniform and uniaxial, and varies with the axial extension.

Problems

1. Obtain the uniaxial *force*–extension relationship for the neo-Hookean bar, and obtain an expression for Young's modulus—the slope of this relationship at $\lambda = 1$—in terms of μ.
2. Show that the foregoing solution is valid in all neo-Hookean cylinders, regardless of section connectedness or shape, if the lateral surface is traction-free.
3. Biological tissues are characterized by a load-bearing microstructure consisting of collagen fibers that are "crimped" in the form of wavy curves in their relaxed state. As the tissue extends, the collagen fibers straighten, or "decrimp," by unbending until they are more-or-less straight; the load required to achieve this is fairly small. Once the decrimp phase is complete, further extension of the tissue requires actual stretching of the collagen fibers. This requires relatively large force compared to that required for decrimping. To model this behavior on the macroscale, we require a strain-energy function which is such that the uniaxial force-extension curve is nearly horizontal for small-to-moderate stretches, while growing rapidly for larger stretches.

   Consider the candidate strain–energy function

$$U = \frac{\mu}{2\gamma}[\exp\{\gamma(I_1 - 3)\} - 1], \tag{7.38}$$

where μ and γ are positive material constants (with μ having dimensions of force/area, while γ is dimensionless). Obtain the force–extension response for an incompressible cylindrical bar subjected to zero traction on its lateral surface, and show that its qualitative properties match the foregoing description of bio-tissue.

4. Consider a cylindrical body occupying the reference configuration defined by $A < R < B$, $-L/2 < Z < L/2$, $0 \leq \theta < 2\pi$. Suppose the cylinder is turned inside out (everted) so that, after deformation, it occupies a new cylindrical region. Thus, the deformation maps the material point with reference position

$$\mathbf{x} = R\mathbf{e}_r(\theta) + Z\mathbf{k} \tag{7.39}$$

to its final position

$$\mathbf{y} = r(R)\mathbf{e}_r(\theta) + z(Z)\mathbf{k}, \tag{7.40}$$

where $a < r < b$ and $z(Z) = -Z$ (i.e., the cross sectional plane $Z = L/2$ in the reference configuration is mapped to the plane $z = -L/2$ in the current configuration, etc.). Also, the inside of the reference cylinder is mapped to the outside of the deformed cylinder, and the outside is mapped to the inside. Thus, $r(A) = b$ and $r(B) = a$.

   (a) Find the function $r(R)$ meeting the stated boundary conditions if the deformation is *isochoric*.

   (b) Compute $\mathbf{C} = \mathbf{F}^t\mathbf{F}$ and obtain $\mathbf{U}$ *by inspection*. Using your result, compute the rotation factor $\mathbf{R}$ in the polar decomposition $\mathbf{F} = \mathbf{RU}$.

   (c) Can this deformation be maintained in equilibrium in an incompressible isotropic material with zero tractions on the lateral surfaces?

7.1.2 *Azimuthal shear*

Imagine a hollow cylinder welded to a rigid shaft at its inner radius, $R = A$, and a rigid cylindrical sleeve at its outer radius, $R = B$. Fix the sleeve and rotate the shaft about its axis $\mathbf{k}$, through the angle Φ. For uniform isotropic incompressible materials, in equilibrium with zero body force, one may feel justified in assuming that an interior circle $R = C$, say, merely rotates uniformly about the shaft without a change in radius and that different concentric circles rotate by different amounts. That is, the azimuth changes by an amount, ϕ say, that depends only on R. To test the hypothesis, we proceed as before to construct the relevant deformation and stress tensors, and then investigate the possibility of satisfying the equation of equilibrium. If Θ is the azimuthal angle prior to deformation we then have

$$\mathbf{x} = R\mathbf{e}_r(\Theta) + Z\mathbf{k} \quad \text{and} \quad \mathbf{y} = R\mathbf{e}_r(\theta) + Z\mathbf{k}, \quad \text{where} \quad \theta = \Theta + \phi(R). \tag{7.41}$$

Then

$$dy = dR\mathbf{e}_r(\theta) + R\mathbf{e}_\theta(\theta)d\theta + dZ\mathbf{k}, \quad \text{where} \quad d\theta = d\Theta + \phi'(R)dR. \tag{7.42}$$

Using eqn (7.4) part 1 with the relevant azimuth yields

$$d\mathbf{y} = \mathbf{e}_r(\theta)[\mathbf{e}_r(\Theta)\cdot d\mathbf{x}] + \mathbf{e}_\theta(\theta)\{[\mathbf{e}_\theta(\Theta) + R\phi'(R)\mathbf{e}_r(\Theta)]\cdot d\mathbf{x}\} + \mathbf{k}[\mathbf{k}\cdot d\mathbf{x}], \tag{7.43}$$

and, hence,

$$\mathbf{F} = \mathbf{e}_r(\theta)\otimes\mathbf{e}_r(\Theta) + \mathbf{e}_\theta(\theta)\otimes\mathbf{e}_\theta(\Theta) + r\phi'(r)\mathbf{e}_\theta(\theta)\otimes\mathbf{e}_r(\Theta) + \mathbf{k}\otimes\mathbf{k}, \tag{7.44}$$

in which $r = R$.

Notice that this may be factored as

$$\mathbf{F} = \hat{\mathbf{F}}\mathbf{Q}, \tag{7.45}$$

where

$$\mathbf{Q} = \mathbf{e}_r(\theta)\otimes\mathbf{e}_r(\Theta) + \mathbf{e}_\theta(\theta)\otimes\mathbf{e}_\theta(\Theta) + \mathbf{k}\otimes\mathbf{k} \tag{7.46}$$

is a rotation and

$$\hat{\mathbf{F}} = \mathbf{I} + \gamma(r)\mathbf{e}_\theta(\theta)\otimes\mathbf{e}_r(\theta) \tag{7.47}$$

is a simple shear on the $\mathbf{e}_r(\theta), \mathbf{e}_\theta(\theta)$ axes of amount $\gamma(r) = r\phi'(r)$ (compare eqn (7.28) part 1). This is an inhomogeneous simple shear. Furthermore, $J = \det\hat{\mathbf{F}}\det\mathbf{Q} = 1$, implying that the deformation is, indeed, isochoric.

Proceeding with the neo-Hookean material for the sake of illustration, we use eqn (7.25) together with $\mathbf{B} = \mathbf{F}\mathbf{F}^t = \hat{\mathbf{F}}\hat{\mathbf{F}}^t$; i.e.,

$$\begin{aligned}\mathbf{B} = {} & \mathbf{e}_r(\theta)\otimes\mathbf{e}_r(\theta) + (1+\gamma^2)\mathbf{e}_\theta(\theta)\otimes\mathbf{e}_\theta(\theta) + \mathbf{k}\otimes\mathbf{k} + \gamma[\mathbf{e}_r(\theta)\otimes\mathbf{e}_\theta(\theta) \\ & + \mathbf{e}_\theta(\theta)\otimes\mathbf{e}_r(\theta)],\end{aligned} \tag{7.48}$$

yielding the stress

$$\begin{aligned}\mathbf{T} = {} & (\mu - p)\mathbf{e}_r(\theta)\otimes\mathbf{e}_r(\theta) + [\mu(1+\gamma^2) - p]\mathbf{e}_\theta(\theta)\otimes\mathbf{e}_\theta(\theta) + (\mu - p)\mathbf{k}\otimes\mathbf{k} \\ & + \mu\gamma[\mathbf{e}_r(\theta)\otimes\mathbf{e}_\theta(\theta) + \mathbf{e}_\theta(\theta)\otimes\mathbf{e}_r(\theta)].\end{aligned} \tag{7.49}$$

Among the three scalar equations of equilibrium, the projection $\mathbf{e}_\theta(\theta)\cdot div\mathbf{T} = 0$ proves to be immediately useful and yields

$$0 = \frac{d}{dr}T_{r\theta} + \frac{2}{r}T_{r\theta} = \frac{1}{r^2}\frac{d}{dr}(r^2 T_{r\theta}), \tag{7.50}$$

where $T_{r\theta}(r) = \mu\gamma(r)$. Thus,

$$\mu r\phi'(r) = \tau/r^2 \tag{7.51}$$

with $\tau = const$. This gives

$$\phi(r) = \Phi + \frac{\tau}{\mu}\int_A^r \frac{dx}{x^3} = \Phi + \frac{\tau}{2\mu}\left(\frac{A^2 - r^2}{r^2A^2}\right). \tag{7.52}$$

The constant τ is determined by imposing the condition $\phi(B) = 0$; thus,

$$-\frac{\tau}{2\mu} = \Phi\frac{A^2B^2}{B^2 - A^2} \tag{7.53}$$

and

$$\phi(r) = \Phi\left[1 - \left(\frac{B}{r}\right)^2\left(\frac{r^2 - A^2}{B^2 - A^2}\right)\right]. \tag{7.54}$$

The deformation is now completely determined.

We have not used the remaining components of the equilibrium equation. We do so now, writing the latter in the form

$$grad p = \mu div\mathbf{B} = \mu\left\{\left[\frac{d}{dr}B_{rr} + \frac{1}{r}(B_{rr} - B_{\theta\theta})\right]\mathbf{e}_r(\theta) + \left(\frac{d}{dr}B_{\theta r} + \frac{2}{r}B_{\theta r}\right)\mathbf{e}_\theta(\theta)\right\}, \tag{7.55}$$

where

$$B_{rr} = 1, \quad B_{rr} - B_{\theta\theta} = -\gamma^2 \quad \text{and} \quad B_{\theta r} = \gamma. \tag{7.56}$$

Combing the last of these with eqn (7.51), we find that

$$\frac{d}{dr}B_{\theta r} + \frac{2}{r}B_{\theta r} = \frac{1}{r^2}\frac{d}{dr}(r^2\gamma) = \frac{1}{r^2}\frac{d}{dr}\left(\frac{\tau}{\mu}\right) = 0. \tag{7.57}$$

Consequently, eqn (7.55) furnishes $\partial p/\partial\theta = 0 = \partial p/\partial z$ and

$$\frac{d}{dr}p(r) = -\mu\gamma^2/r. \tag{7.58}$$

This determines the constraint pressure distribution apart from a constant.

The solution may be used to generate the overall torque-twist relation of the annular cylinder. To see this we compute the traction transmitted by the material to the central shaft. This is

$$\mathbf{T}\mathbf{e}_r(\theta)_{|r=A} = -p(A)\mathbf{e}_r(\theta) + \mu\mathbf{B}\mathbf{e}_r(\theta) = [\mu - p(A)]\mathbf{e}_r(\theta) + (\tau/A^2)\mathbf{e}_\theta(\theta), \tag{7.59}$$

and so the torque, per unit axial length, transmitted *by* the shaft *to* the material is

$$\mathbf{m} = -\int_0^1 \int_0^{2\pi} \mathbf{y}_A \times \{[\mu - p(A)]\mathbf{e}_r(\theta) + (\tau/A^2)\mathbf{e}_\theta(\theta)\}Ad\theta dZ, \tag{7.60}$$

where

$$\mathbf{y}_A = A\mathbf{e}_r(\theta) + Z\mathbf{k} \tag{7.61}$$

is position on the interface between shaft and material. Expanding the cross product, and using the periodicity of $\mathbf{e}_r(\theta)$ and $\mathbf{e}_\theta(\theta)$, we finally derive $\mathbf{m} = m(\Phi)\mathbf{k}$, where

$$m(\Phi) = -2\pi\tau = 4\pi\mu\Phi\left(\frac{A^2B^2}{B^2 - A^2}\right). \tag{7.62}$$

The linearity of this relationship, which is atypical, is an artifact of the linearity of the neo-Hookean response in simple shear. Importantly, this prediction is insensitive to the pressure field, which, as we have seen, is determined apart from a constant. That is, the boundary-value problem, as stated, determines the stress apart from a constant pressure field and, thus, yields a non-unique stress field. To obtain a unique stress, it is necessary to impose one additional scalar condition. One choice is the net axial force transmitted across a cross section.

Problems

1. Consider equilibrium without body force and assume a deformation of the form

$$\mathbf{y} = \mathbf{x} + w(r)\mathbf{k}, \tag{7.63}$$

where $r \in [a, b]$ is the radius from an axis of symmetry prior to deformation.

(a) Show that the deformation gradient is

$$\mathbf{F} = \mathbf{I} + w'(r)\mathbf{k} \otimes \mathbf{e}_r. \tag{7.64}$$

(b) Find $w(r)$ for a neo-Hookean material, assuming the boundary conditions $w(a) = W, w(b) = 0$.

(c) Compute the traction on the inner boundary and determine the allowable range of values of W, if the bond at $r = a$ fails at a critical value of the shear stress.

2. Consider the deformation

$$\mathbf{y} = \mathbf{x} + w(\theta)\mathbf{k}, \tag{7.65}$$

where θ is the azimuthal angle in a cylindrical polar coordinate system in the reference configuration.

(a) Show that the deformation gradient is

$$\mathbf{F} = \mathbf{I} + r^{-1}w'(\theta)\mathbf{k} \otimes \mathbf{e}_\theta. \tag{7.66}$$

(b) Show that the most general function $w(\theta)$ for which the principal invariants $I_{1,2,3}$ of $\mathbf{C}$ are *independent* of θ is of the form

$$w(\theta) = A\theta + B, \tag{7.67}$$

where A and B are constants. Can you interpret this deformation in physical terms?

(c) Using the expression for the stress in an incompressible isotropic material, discuss the problem of maintaining this deformation in equilibrium without body force, using a reference configuration in the form of a right circular cylinder with annular cross section of inner and outer radii, a and b. Specifically, is this deformation possible if the tractions are zero at the inner and outer surfaces? Are there any restrictions on the strain–energy function in this case?

7.1.3 *Torsion of a solid circular cylinder*

In this deformation an entire cross section $Z = const.$ is rotated about the axis of the cylinder, without expansion or contraction, by an amount that depends on the value of Z. We assume this dependence to be linear. Thus,

$$\mathbf{y} = R\mathbf{e}_r(\theta) + Z\mathbf{k}, \quad \text{with} \quad \theta = \Theta + \tau Z, \tag{7.68}$$

where τ is the constant twist, i.e., the rate of rotation with respect to the axial coordinate. Using eqn (7.4), part 1, with the appropriate azimuth and proceeding as before, we derive

$$\mathbf{F} = \mathbf{e}_r(\theta) \otimes \mathbf{e}_r(\Theta) + \mathbf{e}_\theta(\theta) \otimes \mathbf{e}_\theta(\Theta) + r\tau\mathbf{e}_\theta(\theta) \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{k}, \tag{7.69}$$

in which $r = R$.

Once again, this may be factored, this time as

$$\mathbf{F} = \hat{\mathbf{F}}\mathbf{Q}, \tag{7.70}$$

where $\mathbf{Q}$ is the rotation encountered earlier, and

$$\hat{\mathbf{F}} = \mathbf{I} + \gamma(r)\mathbf{e}_\theta(\theta) \otimes \mathbf{k} \tag{7.71}$$

is now an inhomogeneous simple shear on the $\mathbf{e}_\theta(\theta), \mathbf{k}$ axes of amount $\gamma(r) = r\tau$. Again, we have $J = \det \hat{\mathbf{F}} \det \mathbf{Q} = 1$ and the deformation is isochoric. We have

$$\mathbf{B} = \mathbf{e}_r(\theta) \otimes \mathbf{e}_r(\theta) + (1 + \gamma^2)\mathbf{e}_\theta(\theta) \otimes \mathbf{e}_\theta(\theta) + \mathbf{k} \otimes \mathbf{k} + \gamma[\mathbf{k} \otimes \mathbf{e}_\theta(\theta) + \mathbf{e}_\theta(\theta) \otimes \mathbf{k}], \tag{7.72}$$

which we use together with the neo-Hookean model to complete the solution subject to the condition that the lateral surface of the cylinder be free of traction.

Thus, we solve

$$grad p = \mu div \mathbf{B}, \tag{7.73}$$

subject to

$$\mathbf{T}\mathbf{e}_r(\theta) = \mathbf{0} \quad \text{at} \quad r = A. \tag{7.74}$$

Notice that for eqn (7.73) to have a solution, it is necessary that the right-hand side have zero curl, i.e., $curl(div\mathbf{B}) = \mathbf{0}$. For general incompressible materials, this condition is replaced by $curl(div\tilde{\mathbf{T}}) = \mathbf{0}$, where $\tilde{\mathbf{T}}$ is the constitutive part of the stress. This, in turn, imposes restrictions on the deformation without regard to the reactive constraint pressure field. The pressure field may then be determined *post facto*, in principle, by path integration. The zero curl condition ensures that the result obtained is independent of the path and, hence, a well-defined function of $\mathbf{x}$.

Returning to the problem at hand, we have

$$div\mathbf{B} = \left[\frac{d}{dr}B_{rr} + \frac{1}{r}(B_{rr} - B_{\theta\theta})\right]\mathbf{e}_r(\theta) = -r\tau^2\mathbf{e}_r(\theta), \tag{7.75}$$

yielding $\frac{d}{dr}p = -\mu r\tau^2$ and, hence,

$$p(r) = p_0 - \frac{1}{2}\mu\tau^2 r^2, \tag{7.76}$$

where $p_0 = p(0)$. We have succeeded here, as well as in the previous examples, in generating the pressure field (apart from a constant) because the curl condition is automatically satisfied. The Cauchy stress is

$$\mathbf{T} = \left(\frac{1}{2}\mu\tau^2 r^2 - p_0\right)\mathbf{I} + \mu\mathbf{B}, \tag{7.77}$$

yielding

$$\mathbf{T}\mathbf{e}_r(\theta) = \left(\frac{1}{2}\mu\tau^2 r^2 - p_0 + \mu\right)\mathbf{e}_r(\theta). \tag{7.78}$$

The constant p_0 is obtained by imposing eqn (7.74), yielding the unique stress field

$$\mathbf{T} = \mu\left[\frac{1}{2}\tau^2(r^2 - A^2) - 1\right]\mathbf{I} + \mu\mathbf{B}. \tag{7.79}$$

In linear elasticity, terms that are nonlinear in τ are neglected. Doing so here, we find that $\mathbf{T} \simeq \mathbf{T}_{lin}$, where

$$\mathbf{T}_{lin} = \mu r\tau[\mathbf{k} \otimes \mathbf{e}_\theta(\theta) + \mathbf{e}_\theta(\theta) \otimes \mathbf{k}], \tag{7.80}$$

which, of course, generates the classical linear shear stress distribution over a cross section. This distribution persists in the nonlinear case, but now normal stresses also arise in response to the twist. This, of course, is just the usual normal stress effect in disguise.

The overall response of the cylinder may be determined by computing the net force on a cross section and the net twisting moment required to effect the twist. These, in turn, require the traction

$$\mathbf{T}\mathbf{k} = \frac{1}{2}\mu\tau^2(r^2 - A^2)\mathbf{k} + \mu r\tau\,\mathbf{e}_\theta(\theta) \tag{7.81}$$

acting on a cross section. The resultant force is

$$\mathbf{f} = \int_0^{2\pi}\int_0^A \mathbf{T}\mathbf{k}r\,dr\,d\theta = f(\tau)\mathbf{k}, \tag{7.82}$$

where

$$f(\tau) = -\frac{1}{4}\pi A^4\mu\tau^2. \tag{7.83}$$

Evidently the force is a manifestation of the normal stress effect, vanishing in the linear theory.

Finally, the twisting moment is

$$\mathbf{m} = \int_0^{2\pi}\int_0^A \mathbf{y} \times \mathbf{T}\mathbf{k}r\,dr\,d\theta = m(\tau)\mathbf{k}, \tag{7.84}$$

where

$$m(\tau) = \frac{1}{2}A^4\mu\tau. \tag{7.85}$$

This is precisely the same result predicted by linear elasticity, the coincidence again being due to the peculiar (i.e., linear) behavior of the neo-Hookean material in simple shear.

Problems

1. Verify the formulas for the net force and twisting moment.
2. Show that a straight generator of the lateral surface of the cylinder is deformed into a helix. Find the ratio of its final length to its initial length.

7.1.4 *Combined extension and torsion*

The one surprising aspect of the torsion problem, at least for those not previously aware of the normal stress effect, is the prediction of a compressive axial force accompanying twist. This is to be regarded as the reaction force supplied by plates welded to the cross sections at the ends of the bar, arising in response to the restriction that the perpendicular distance between the plates (the end-to-end length of the bar) remains fixed. This suggests that if the reaction force is relaxed, then the end-to-end length should adjust accordingly. To investigate this possibility, we propose a simple adjustment of the foregoing kinematics to accommodate axial extension. Thus, in place of eqn (7.68), we consider the deformation

$$\mathbf{y} = r(R)\mathbf{e}_r(\theta) + z\mathbf{k}, \quad \text{with} \quad \theta = \Theta + \psi z \quad \text{and} \quad z = \lambda Z \tag{7.86}$$

where ψ is the constant twist. Here, the rate of rotation with respect to axial length on the *deformed* cylinder, and λ is a positive constant. We allow r to be unequal to R to accommodate incompressibility; the cross section must adjust to the axial stretch so as to preserve volume.

The usual procedure generates

$$\mathbf{F} = r'(R)\mathbf{e}_r(\theta) \otimes \mathbf{e}_r(\Theta) + (r/R)\mathbf{e}_\theta(\theta) \otimes \mathbf{e}_\theta(\Theta) + r\lambda\psi\mathbf{e}_\theta(\theta) \otimes \mathbf{k} + \lambda\mathbf{k} \otimes \mathbf{k}. \tag{7.87}$$

Incompressibility is not automatic this time; to enforce it, we compute the determinant of $\mathbf{F}$ in terms of the given parameters and set it to unity. The easiest way to proceed is to use the scalar triple product, or box product, definition of the determinant. Readers unfamiliar with this should consult the excellent discussion in Chadwick (1976). Thus,

$$J = [\mathbf{F}\mathbf{e}_r(\Theta), \mathbf{F}\mathbf{e}_\theta(\Theta), \mathbf{F}\mathbf{k}] = [r'(R)\mathbf{e}_r(\theta), (r/R)\mathbf{e}_\theta(\theta), r\lambda\psi\mathbf{e}_\theta(\theta) + \lambda\mathbf{k}] = \lambda(r/R)r'(R), \tag{7.88}$$

in which the square brackets are used to denote the box product. This is the same as eqn (7.33), and carries the same result; namely, eqn (7.34) in the case of an isochoric deformation and a solid section with $r(0) = 0$. Inserting the latter into eqn (7.87), we arrive at

$$\begin{aligned} \mathbf{B} = {} & \lambda^{-1}\mathbf{e}_r(\theta) \otimes \mathbf{e}_r(\theta) + (\lambda^{-1} + r^2\psi^2\lambda^2)\mathbf{e}_\theta(\theta) \otimes \mathbf{e}_\theta(\theta) \\ & + \lambda^2\mathbf{k} \otimes \mathbf{k} + r\psi\lambda^2[\mathbf{k} \otimes \mathbf{e}_\theta(\theta) + \mathbf{e}_\theta(\theta) \otimes \mathbf{k}]. \end{aligned} \tag{7.89}$$

For the neo-Hookean material we use eqn (7.73), obtaining

$$grad p = \frac{\mu}{r}(B_{rr} - B_{\theta\theta})\mathbf{e}_r(\theta), \tag{7.90}$$

implying once again that p depends only on r, with derivative

$$\frac{d}{dr}p = -\mu r\psi^2\lambda^2. \tag{7.91}$$

We then have

$$p(r) = p_a + \frac{\mu}{2}\psi^2\lambda^2(a^2 - r^2), \quad \text{where} \quad p_a = p(a) \quad \text{and} \quad a = A/\sqrt{\lambda}. \tag{7.92}$$

Suppose, again, that the traction vanishes at the lateral surface. Then, $\mathbf{T}\mathbf{e}_r(\theta)$ vanishes at $r = a$, yielding $p_a = \mu/\lambda$ and, hence, the unique pressure field

$$p(r) = \mu\left[\lambda^{-1} + \frac{1}{2}\psi^2\lambda^2(a^2 - r^2)\right] \tag{7.93}$$

and with this the stress field is completely specified.

The traction on a cross section is now given by

$$\mathbf{T}\mathbf{k} = (\mu\lambda^2 - p)\mathbf{k} + \mu r\psi\lambda^2\mathbf{e}_\theta(\theta) \tag{7.94}$$

and generates the twisting moment $\mathbf{m} = m\mathbf{k}$, where (compare eqn (7.85))

$$m = \frac{1}{2}\pi\mu a^4\psi\lambda^2 = \frac{1}{2}\pi\mu A^4\psi \tag{7.95}$$

whereas the net force is $\mathbf{f} = f\mathbf{k}$, where

$$f = \pi\mu a^2\left(\lambda^2 - \lambda^{-1} - \frac{1}{4}\psi^2\lambda^2 a^2\right). \tag{7.96}$$

Relaxing the force has the effect of coupling the extension to the twist, resulting in

$$\lambda^2 - \lambda^{-1} = \frac{1}{4}\psi^2\lambda^2 a^2, \tag{7.97}$$

which, in turn, requires that $\lambda > 1$ whenever $\psi \neq 0$. Thus, the bar extends as it is twisted. This is, again, just the normal stress effect, variously referred to as the Swift effect or the Poynting effect, depending on the context. The prediction that extension of a bar is induced by a twisting moment is corroborated by experiments.

It bears mentioning that we have said nothing about the stability of these equilibria. In practice, torsional buckling ensues if the twist is large, yielding the possibility of an alternative deformation of a slender bar into a helical shape.

7.2 Problems exhibiting radial symmetry with respect to a fixed point

In this class of problems, the distance of material points from a specified origin changes, but nothing else, while all points lying on a sphere centered at the origin move radially by the same amount. Thus,

$$\mathbf{y} = \lambda(R)\mathbf{x}, \quad \text{where} \quad R = |\mathbf{x}| \tag{7.98}$$

is the distance from the origin prior to deformation. The distance after is simply the radius

$$r(R) = R\lambda(R), \tag{7.99}$$

yielding λ as the ratio of the radii, also known as the hoop stretch.

To obtain the deformation gradient, consider

$$d\mathbf{y} = \lambda'(R)\mathbf{x}dR + \lambda(R)d\mathbf{x}. \tag{7.100}$$

As usual, we want this as a linear function of $d\mathbf{x}$, so that we can read off the desired result. To this end, we differentiate $R^2 = \mathbf{x} \cdot \mathbf{x}$, obtaining $dR = \mathbf{u} \cdot d\mathbf{x}$, where $\mathbf{u} = R^{-1}\mathbf{x}$ is the normalized radius vector; hence,

$$\mathbf{F} = R^{-1}\lambda'(R)\mathbf{x} \otimes \mathbf{x} + \lambda(R)\mathbf{I} = r'(R)\mathbf{u} \otimes \mathbf{u} + \lambda(R)(\mathbf{I} - \mathbf{u} \otimes \mathbf{u}), \tag{7.101}$$

where $r'(R) = R\lambda'(R) + \lambda(R)$. We then have

$$J = \lambda^2 r' \tag{7.102}$$

and thus require, as before, that $r(R)$ be an increasing function; i.e., $r'(R) > 0$. The polar decomposition is trivial in this case, yielding $\mathbf{R} = \mathbf{I}$, $\mathbf{U} = \mathbf{F}$, and the principal stretches

$$\{\lambda_i\} = \{r', \lambda, \lambda\}. \tag{7.103}$$

Suppose the material is incompressible and the deformation, therefore, isochoric. Putting $J = 1$ in eqn (7.102) yields a simple differential equation, having the unsurprising solution

$$r^3 - a^3 = R^3 - A^3, \tag{7.104}$$

where $a = r(A)$. Then, the hoop stretch distribution is

$$\lambda(R) = \left(1 + \frac{a^3 - A^3}{R^3}\right)^{1/3}, \tag{7.105}$$

and the radial stretch is $\lambda_1 = \lambda^{-2}$.

For isotropic materials we combine eqns (4.41) and (6.23), with $J = 1$, obtaining

$$\mathbf{T} = \sum \tau_i \mathbf{v}_i \otimes \mathbf{v}_i - p\mathbf{I}, \tag{7.106}$$

in which

$$\tau_1 = \lambda_1 \partial\omega/\partial\lambda_1, \quad \text{etc.}, \tag{7.107}$$

and, for the present class of deformations, $\lambda_2 = \lambda_3 (= \lambda)$; consequently, $\tau_2 = \tau_3$. Using $\mathbf{v}_1 = \mathbf{u}$ thus yields

$$\mathbf{T} = \tau_1 \mathbf{u} \otimes \mathbf{u} + \tau_2(\mathbf{I} - \mathbf{u} \otimes \mathbf{u}) - p\mathbf{I}, \tag{7.108}$$

which, in the present circumstances, may be written, for uniform materials, in the form

$$\mathbf{T} = h(r)\mathbf{y} \otimes \mathbf{y} - [p - g(r)]\mathbf{I}, \tag{7.109}$$

with

$$h(r) = r^{-2}f(r), \quad f(r) = \tau_1 - \tau_2 \quad \text{and} \quad g(r) = \tau_2. \tag{7.110}$$

Then, for equilibrium in the absence of body forces,

$$\mathbf{0} = div\mathbf{T} = div(h\mathbf{y} \otimes \mathbf{y}) - grad(p - g). \tag{7.111}$$

Of course, no one can remember the formula for the divergence in spherical coordinates, and so we will use Cartesians instead, i.e., $div\mathbf{T} = T_{ij,j}\mathbf{e}_i$, where $T_{ij,j} = \partial T_{ij}/\partial y_j$, yielding

$$(hy_iy_j)_{,j} - p_{,i} + g_{,i} = 0. \tag{7.112}$$

Expanding this using $y_{i,j} = \delta_{ij}$ (the Kronecker delta), $y_{j,j} = 3$ and $(\cdot)_{,j} = r^{-1}(\cdot)'y_j$, for any function of r alone, yields

$$grad p = [4h + rh'(r) + r^{-1}g'(r)]\mathbf{y}. \tag{7.113}$$

This is enough to conclude that p also depends on r alone, with derivative

$$p'(r) = r[4h + rh'(r) + r^{-1}g'(r)] \tag{7.114}$$

Integration and application of suitable boundary conditions thus determines the solution.

Now that we know the constraint pressure depends only on radius, we may re-write eqn (7.109) in the form

$$\mathbf{T} = F(r)\mathbf{u} \otimes \mathbf{u} + G(r)\mathbf{I} = H(r)\mathbf{y} \otimes \mathbf{y} + G(r)\mathbf{I}, \tag{7.115}$$

where

$$F(r) = t_1 - t_2, \quad G(r) = t_2, \quad H(r) = r^{-2}F(r) \tag{7.116}$$

and

$$t_1 = \tau_1 - p, \quad t_2 = \tau_2 - p(= t_3) \tag{7.117}$$

are the (principal) radial and hoop components of the Cauchy stress. Noting that eqn (7.115) resembles eqn (7.109), we proceed immediately to obtain the equilibrium equation

$$4H + rH'(r) + r^{-1}G'(r) = 0. \tag{7.118}$$

This is easily seen to be equivalent to

$$(F + G)' + 2r^{-1}F = 0, \tag{7.119}$$

which may be converted to the more recognizable form

$$\frac{d}{dr}t_1 + \frac{2}{r}(t_1 - t_2) = 0. \tag{7.120}$$

There is just one non-trivial equilibrium equation to be solved for the single unknown function $\lambda(R)$.

7.2.1 *Integration of the equation*

Actually, it proves convenient to use stretch as the independent variable. First, we define

$$\hat{\omega}(\lambda) = \omega(\lambda^{-2}, \lambda, \lambda). \tag{7.121}$$

Using the chain rule,

$$\begin{aligned}\hat{\omega}'(\lambda) &= (\partial\omega/\partial\lambda_1)d\lambda_1/d\lambda + (\partial\omega/\partial\lambda_2)d\lambda_2/d\lambda + (\partial\omega/\partial\lambda_3)d\lambda_3/d\lambda \\ &= 2\partial\omega/\partial\lambda_2 - 2\lambda^{-3}\partial\omega/\partial\lambda_1,\end{aligned} \tag{7.122}$$

we find that

$$\begin{aligned}t_1 - t_2 &= \lambda_1\partial\omega/\partial\lambda_1 - \lambda_2\partial\omega/\partial\lambda_2 \\ &= \lambda^{-2}\partial\omega/\partial\lambda_1 - \lambda\left[\frac{1}{2}\hat{\omega}'(\lambda) + \lambda^{-3}\partial\omega/\partial\lambda_1\right] \\ &= -\lambda\frac{1}{2}\hat{\omega}'(\lambda).\end{aligned} \tag{7.123}$$

Equation (7.120) then yields

$$\frac{d}{dr}t_1 = \frac{\lambda}{r}\hat{\omega}'(\lambda) = R^{-1}\hat{\omega}'(\lambda), \tag{7.124}$$

which may be reduced, using

$$\frac{d}{dr}t_1 = \frac{1}{\lambda_1}\frac{d}{dR}t_1 = \lambda^2\frac{d}{dR}t_1, \tag{7.125}$$

to

$$R\lambda^2\frac{d}{dR}t_1 = \hat{\omega}'(\lambda). \tag{7.126}$$

To convert the derivative on the left, we use eqn (7.105), reaching

$$R\lambda'(R) = -\frac{(\lambda^3 - 1)}{\lambda^2}. \tag{7.127}$$

We then use this with (7.126) to obtain

$$\frac{d}{d\lambda}t_1 = -\frac{\hat{\omega}'(\lambda)}{\lambda^3 - 1}, \tag{7.128}$$

and, thus, reduce the problem to a quadrature.

7.2.2 *Pressurized shells, cavitation*

Consider, for example, a spherical shell $A \leq R \leq B$, traction-free at the inner radius, and subject to a negative pressure or suction at the outer radius. This simulates a triaxial state of stress in a region of material surrounding a spherical hole. The boundary conditions are of the form

$$t_1\mathbf{u} = \mathbf{T}\mathbf{u} = P\mathbf{u}, \tag{7.129}$$

in which $-P$ is the assigned pressure, and therefore,

$$t_1 = P \quad \text{at} \quad \lambda = \lambda_b \quad \text{and} \quad t_1 = 0 \quad \text{at} \quad \lambda = \lambda_a, \tag{7.130}$$

where $\lambda_a = a/A$ and $\lambda_b = b/B$, with $b^3 - a^3 = B^3 - A^3$, are the hoop stretches at the inner and outer radii. Taken together with eqn (7.128), this furnishes

$$P = \int_{\lambda_b}^{\lambda_a} \frac{\hat{\omega}'(\lambda)}{\lambda^3 - 1} d\lambda, \tag{7.131}$$

yielding P vs a (or b) once the strain-energy function is specified.

This seemingly innocuous result may be used to furnish a graphic illustration of the power of nonlinear elasticity to predict dramatic phenomena. Consider the case of a solid sphere, $A = 0$, and suppose the sphere remains solid, no matter the suction, ($a = 0$). Then $\lambda(R) = 1$, $\lambda_{a,b} = 1$, $\mathbf{F} = \mathbf{I}$ and $\mathbf{T}$ is of the form $\mathbf{T} = -p\mathbf{I}$, in which p is uniform in equilibrium without body force. Then, unsurprisingly, $P(= -p)$ is indeterminate; the rubber remains undeformed no matter the suction. This state, therefore, furnishes a solution for all values of suction.

Experimental evidence (see Gent and Lindley, 1958) suggests that a hole forms spontaneously at the center of the sphere when the suction is sufficiently strong. This *cavitation* solution corresponds to $a > 0$, where $a = r(0)$. In this case λ_a is unbounded and the critical suction for its sudden onset is

$$P_{crit} = \int_1^{\infty} \frac{\hat{\omega}'(\lambda)}{\lambda^3 - 1} d\lambda. \tag{7.132}$$

The lower limit is explained by observing that

$$\lambda_b = b/B = (1 + a^3/B^3)^{1/3}, \tag{7.133}$$

and thus that $\lambda_b \to 1$ as $a \to 0$. Of course, all this is sensible only if P_{crit} is finite, or in other words, if the integral in eqn (7.132) exists. This may or may not be the case, depending on the strain–energy function at hand. In the latter case, we conclude that cavitation is not feasible and, hence, that the trivial solution is the only one available in the class of deformations considered. Ironically, strain–energy functions are deduced, traditionally, on the basis of experiments involving finite stretches, and so the theoretical study of cavitation using the present solution requires knowledge of material response over a far wider range of deformation than is normally encountered in experiments designed to quantify material response. Indeed, a rubber band breaks at a fairly moderate value of uniform overall stretch. However, having said this it must also be noted that rupture is invariably accompanied by strongly inhomogeneous deformations that may include cavitation on the micro-scale! All of this is food for thought as one contemplates theory and supporting experiments for failure mechanisms in rubber.

The post-cavitation response is given simply by

$$P(a) = \int_{\lambda_b}^{\infty} \frac{\hat{\omega}'(\lambda)}{\lambda^3 - 1} d\lambda, \tag{7.134}$$

where λ_b is given by eqn (7.133). This bifurcates off the trivial solution at $P = P_{crit}$.

Problems

1. Repeat the foregoing for the simpler case of plane strain, i.e., for the two-dimensional radial expansion of a cylinder.

   In **problems 2–5** assume the material to be incompressible, isotropic, and neo-Hookean.

2. Consider the eversion of an incompressible hemispherical shell. Assume the deformation is such that the final radius depends only on the initial radius, and that the elevation angle above the equator is mapped to its opposite value, below the equator. Show that equilibrium cannot be maintained with vanishing tractions at the inner and outer constant-radius surfaces. The actual deformation entails a flaring of the shell as required to meet the zero-traction conditions.

3. Find the critical negative pressure P_{crit} for the onset of cavitation of a solid sphere ($A = 0$). What is the relation between the negative pressure and the cavity radius $a = r(0)$?

4. A solid circular bar has initial radius A and length L. Suppose the bar has density ρ and let it spin about its own axis at the constant rate ω. This spin causes a contraction of the bar along its axis. Let $\mathbf{u}(x) = \cos x\mathbf{e}_1 + \sin x\mathbf{e}_2$. The corresponding

deformation is described by $\mathbf{x} = R\mathbf{u}(\Theta) + Z\mathbf{k}$ and $\mathbf{y} = r(R)\mathbf{u}(\theta) + z\mathbf{k}$, where $\theta = \Theta + \omega t$, $z = \lambda Z$, and λ is the constant stretch along the axis. Suppose the traction is zero on the lateral surface of the cylinder.

(a) Calculate the resultant forces $\pm f\mathbf{k}$ on the two ends of the cylinder. Find the value of λ corresponding to $f = 0$.

(b) Obtain a relation between the deformed length of the cylinder and ω.

5. The kinematics of pure flexure of a block are described by

$$\mathbf{x} = x_A\mathbf{E}_A, \quad \mathbf{y} = r\mathbf{e}_r(\theta) + z\mathbf{k}. \tag{7.135}$$

Here, $r = f(x_1)$ and $\theta = g(x_2)$, for some functions f and g to be determined. Thus, straight lines $x_1 = const.$ and $x_2 = const.$ are mapped to concentric circular arcs and rays through the origin, respectively (draw a figure). Furthermore, $\mathbf{k} = \mathbf{e}_3$ and $z = x_3$, so the deformation is a plane strain (take $\{\mathbf{e}_i\} = \{\mathbf{E}_A\}$).

The reference configuration is the region defined by $A_1 \leq x_1 \leq A_2$, $-B \leq x_2 \leq B$, $-H \leq x_3 \leq H$. Suppose there are no tractions applied to the edges $x_1 = A_1, A_2$. The neutral axis is defined to be the vertical line $x_1 = x_1^{n}$ that neither lengthens nor shortens in the course of deformation. Find $r_n = f(x_1^{n})$, the radius of curvature of the neutral axis in the deformed configuration. Let $a_1 = f(A_1)$, $a_2 = f(A_2)$ and find a relationship involving a_1, a_2 and r_n.

Show that the resultant forces on the edges $x_2 = \pm B$ vanish. Calculate the resultant moments of the traction distributions on the edges. What is the relation between the moment and the curvature $\kappa_n = 1/r_n$ of the neutral axis?

6. Consider the homogeneously deforming unit cube of Problem 3 in Chapter 6, but now suppose that it is subjected to equi-biaxial loading. Thus, the traction vanishes on the faces with unit normals $\pm\mathbf{e}_3$, while the forces on the faces with unit normals $\pm\mathbf{e}_1$ and $\pm\mathbf{e}_2$ are $\pm f\mathbf{e}_1$ and $\pm f\mathbf{e}_2$, respectively, where $f > 0$.

(a) Show that a solution with $\lambda_1 = \lambda_2$ exists for all such f.

(b) Using the so-called Mooney–Rivlin strain–energy function defined by $W = C_1(I_1 - 3) + C_2(I_2 - 3)$, where C_1 and C_2 are given material constants, show that another branch of solutions, with $\lambda_1 \neq \lambda_2$, becomes possible when f reaches a critical value. Thus, there is a bifurcation of equilibria at this value, at which the solution with equi-biaxial stretch bifurcates to one with unequal stretches. This behavior has been observed experimentally and has come to be known as the Treloar–Kearsely instability. We will study the stability of these solutions later in the course.

(c) Our block is isotropic, by assumption, with respect to its initial configuration (the unit cube), which we have chosen as reference. Consider a deformation characterized by unequal biaxial stretch (i.e., $\lambda_1 \neq \lambda_2$, $\lambda_3 = 1/\lambda_1\lambda_2$). What is the symmetry group relative to this deformed configuration? Is the material isotropic relative to this configuration?

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8

Some examples involving uniform, compressible isotropic materials

We study some examples of deformation in unconstrained isotropic materials. As the simplification afforded by an *a priori* constraint on the deformation is not available, we confine attention to strain–energy functions that facilitate analytical treatment.

8.1 Spherical symmetry, revisited

Recall the kinematical development in eqns (7.98)–(7.102) for deformations having a center of symmetry, but this time do not impose incompressibility. We adopt the constitutive formulation developed in eqns (4.56) and (4.57), which yields

$$\mathbf{P} = (w_1 + i_2 w_2)\mathbf{I} - w_2\mathbf{F} + w_3\mathbf{F}^* \tag{8.1}$$

in the present circumstances, and hence,

$$Div\mathbf{P} = \nabla(w_1 + i_2 w_2) + \mathbf{F}^*(\nabla w_3) - Div(w_2\mathbf{F}), \tag{8.2}$$

where use has been made of the Piola identity $Div\mathbf{F}^* = \mathbf{0}$.

Using eqn (7.101), we find that

$$\mathbf{F}^* = \lambda^2\mathbf{u}\otimes\mathbf{u} + \lambda r'(\mathbf{I} - \mathbf{u}\otimes\mathbf{u}) \tag{8.3}$$

and, for uniform materials,

$$\mathbf{F}^*(\nabla w_3) = (w_3)'\mathbf{F}^*\mathbf{u} = \lambda^2(w_3)'\mathbf{u}, \tag{8.4}$$

where we have made use of $\nabla R = \mathbf{u}$. Furthermore,

$$\nabla(w_1 + i_2 w_2) = (w_1 + i_2 w_2)'\mathbf{u}. \tag{8.5}$$

Consider a strain-energy function having $w_2 = 0$. In this case the equilibrium equation reduces to

$$[(w_1)' + \lambda^2(w_3)']\mathbf{u} = \mathbf{0}, \tag{8.6}$$

yielding the ordinary differential equation:

$$(w_1)' + \lambda^2(w_3)' = 0. \tag{8.7}$$

As an example, consider the class of compressible Varga-type materials defined by

$$w(i_1, i_2, i_3) = 2\mu[i_1 + F(i_3)], \tag{8.8}$$

where μ is a positive material constant. This is simply the linear shear modulus, as in the case of conventional Varga materials. We then have $w_1 = 2\mu$, a constant, and, $w_3 = 2\mu F'(i_3)$. The differential eqn (8.7) simplifies to $F''(i_3)i_3'(R) = 0$. Assuming $F''(i_3) \neq 0$, recalling that $i_3 = J$ and using eqn (7.102), we obtain

$$r^2 r'(R) = JR^2, \quad \text{with} \quad J = const. \tag{8.9}$$

Integrating and imposing $r(A) = \lambda_A A$, where λ_A is an assigned positive constant, we finally obtain the deformation

$$r(R)^3 = JR^3 + (\lambda_A^3 - J)A^3. \tag{8.10}$$

Here A can be identified with the initial radius of a sphere, and λ_A and the ratio of final to initial sphere radii.

It remains to determine the constant J. For example, in the case of a solid sphere, it would be natural to require that $r(0) = 0$, corresponding to another solid sphere. In this case, we find $J = \lambda_A^3$ and $r(R) = \lambda_A R$. The deformation gradient is $\mathbf{F} = \lambda_A \mathbf{I}$, a uniform equi-triaxial stretch. We refer to this as the trivial solution.

To explore conditions under which cavitation is possible, we consider the case $r(0) = a > 0$, for some constant a. Evidently, this requires $J < \lambda_A^3$. Furthermore, if the newly-created hole is traction free, then we must impose $\mathbf{Tu} = \mathbf{0}$ at $r = a$, where

$$\mathbf{Tu} = \lambda^{-2}\partial\omega/\partial\lambda_1\mathbf{u}. \tag{8.11}$$

Accordingly, we impose

$$\lambda^{-2} + F'(J) = 0 \tag{8.12}$$

at $r = a$. Because $\lambda \to \infty$ as $r \to a(R \to 0)$ in this case, we require J to be such that

$$F'(J) = 0. \tag{8.13}$$

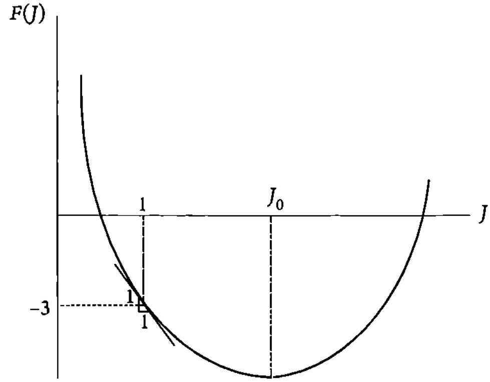

Figure 8.1 Constitutive response of a Varga-type material capable of supporting cavitation

Accordingly, if a solution exhibiting cavitation is to exist, the function F must have at least one stationary point, J_0, say. Such a function, adjusted to ensure that the strain energy and Cauchy stress vanish when the material is undeformed, is sketched in Figure 8.1. The cavitated solution is then given by

$$r(R)^3 = J_0R^3 + (\lambda_A^3 - J_0)A^3, \tag{8.14}$$

and is available provided that the boundary displacement is such that $\lambda_A > J_0^{1/3}$.

In this solution and in the trivial solution, the deformation is controlled entirely by λ_A. To choose between them, we compare the total energies required to maintain the two solutions. In the case of any spherically symmetric deformation this is given by

$$E(\lambda_A) = 4\pi \int_0^A wR^2 dR, \tag{8.15}$$

where

$$\begin{aligned} w/2\mu &= i_1 + F(J) \\ &= r'(R) + 2r/R + F(J) \\ &= R^{-2}(R^2r)' + F(J). \end{aligned} \tag{8.16}$$

We have

$$\int_0^A R^{-2}(R^2r)'R^2dR = A^2r(A) = \lambda_A A^3, \tag{8.17}$$

which is fixed by the data, and hence the energy comparison

$$E(\lambda_A) - E_{cav}(\lambda_A) = 4\pi \int_0^A R^2[F(J) - F(J_0)]dR, \tag{8.18}$$

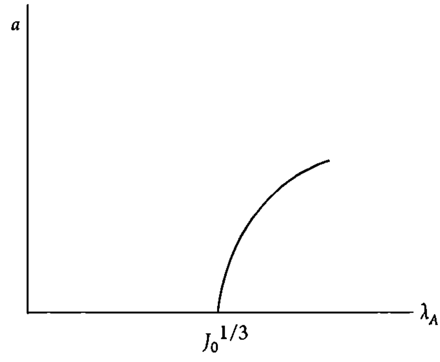

Figure 8.2 Cavitated solution bifurcates off trivial solution at $\lambda_A = J_0^{1/3}$

where E_{cav} is the energy of the cavitated equilibrium solution. Of course, this is meaningful only if $\lambda_A > J_0^{1/3}$. For the kind of material sketched in Figure 8.1, J_0 furnishes the minimum of the function F, ensuring that $E_{cav}(\lambda_A) \leq E(\lambda_A)$; the cavitated equilibrium deformation thus requires less energy than any alternative spherically symmetric deformation, including the trivial equilibrium deformation. However, we have not proved that it minimizes the energy relative to any kinematically possible (non-spherically symmetric) deformation. Nevertheless, the analysis provides support for the conclusion that cavitation emerges when the boundary radius exceeds the critical value $J_0^{1/3}$. The cavity radius is $a = (\lambda_A^3 - J_0)^{1/3}A$ (see Figure 8.2).

8.2 Plane strain

The term *plane strain* is used in reference to the two-dimensional situation:

$$\mathbf{x} = \mathbf{x}_{\|} + z\mathbf{k}, \quad \mathbf{y} = \mathbf{y}_{\|} + z\mathbf{k} \quad \text{with} \quad \mathbf{y}_{\|} = \chi_{\|}(\mathbf{x}_{\|}), \tag{8.19}$$

where $\mathbf{k}$ is a unit normal to a fixed plane Ω in which the deformation occurs, containing $\mathbf{x}_{\|}$ and $\mathbf{y}_{\|}$. The associated deformation gradient is of the form

$$\mathbf{F} = \mathbf{F}_{\|} + \mathbf{k} \otimes \mathbf{k}, \tag{8.20}$$

wherein $\mathbf{F}_{\|}$ maps Ω to itself. This may be written

$$\mathbf{F}_{\|} = \sum_{\alpha=1}^{2} \lambda_\alpha \mathbf{v}_\alpha \otimes \mathbf{u}_\alpha, \tag{8.21}$$

where the λ_α are the principal stretches ($\lambda_3 = 1$) and $\{\mathbf{v}_\alpha\}$, $\{\mathbf{u}_\alpha\}$ are orthonormal principal strain axes in Ω ($\mathbf{u}_3 = \mathbf{v}_3 = \mathbf{k}$). The former are the roots of the quadratic characteristic equation

$$\lambda^2 - I\lambda + J = 0, \tag{8.22}$$

where

$$I = \lambda_1 + \lambda_2 = tr\mathbf{U}_{\|}, \quad J = \lambda_1\lambda_2 = \det \mathbf{U}_{\|}, \tag{8.23}$$

and $\mathbf{U}_{\|}$ is the right stretch factor in the polar decomposition of $\mathbf{F}_{\|}$, i.e., $\mathbf{U}_{\|} = \sum_{\alpha=1}^{2} \lambda_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha$. The rotation factor is $\mathbf{R}_{\|} = \sum_{\alpha=1}^{2} \mathbf{v}_\alpha \otimes \mathbf{u}_\alpha$ and the cofactor is $\mathbf{F}_{\|}^{*} = \lambda_2 \mathbf{v}_1 \otimes \mathbf{u}_1 + \lambda_1 \mathbf{v}_2 \otimes \mathbf{u}_2$.

The stretches are determined by I and J, implying that the strain energy for isotropic materials in a plane–strain deformation is

$$\omega(\lambda_1, \lambda_2, 1) = w(I, J), \tag{8.24}$$

for some function w. This furnishes

$$\partial\omega/\partial\lambda_1 = w_I + \lambda_2 w_J, \quad \partial\omega/\partial\lambda_2 = w_I + \lambda_1 w_J \tag{8.25}$$

whereas $\partial\omega/\partial\lambda_3$, evaluated at $\lambda_3 = 1$, is a function of the λ_α and, hence, a function of $\mathbf{x}_{\|}$. The Piola stress reduces to

$$\mathbf{P} = \mathbf{P}_{\|} + \partial\omega/\partial\lambda_3 \mathbf{k} \otimes \mathbf{k}, \tag{8.26}$$

where

$$\begin{aligned}\mathbf{P}_{\|} &= w_I(\mathbf{v}_1 \otimes \mathbf{u}_1 + \mathbf{v}_2 \otimes \mathbf{u}_2) + w_J(\lambda_2 \mathbf{v}_1 \otimes \mathbf{u}_1 + \lambda_1 \mathbf{v}_2 \otimes \mathbf{u}_2) \\ &= w_I \mathbf{R}_{\|} + w_J \mathbf{F}_{\|}^{*}.\end{aligned} \tag{8.27}$$

For general applications, it is useful to observe that

$$\mathbf{F}_{\|} + \mathbf{F}_{\|}^{*} = (\lambda_1 + \lambda_2)(\mathbf{v}_1 \otimes \mathbf{u}_1 + \mathbf{v}_2 \otimes \mathbf{u}_2) = I\mathbf{R}_{|} \tag{8.28}$$

and, hence, that

$$\mathbf{P}_{\|} = I^{-1} w_I(\mathbf{F}_{\|} + \mathbf{F}_{\|}^{*}) + w_J \mathbf{F}_{\|}^{*} \tag{8.29}$$

in any plane–strain deformation.

Problems

1. (a) In three dimensions, establish the polar decomposition

$$\mathbf{F} = \mathbf{RU} \tag{8.30}$$

in which $\mathbf{R} \in Orth^{+}$ and

$$\begin{aligned}\mathbf{U} &= \sum \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i \\ &= (\lambda \mathbf{I})(s\mathbf{u}_1 \otimes \mathbf{u}_1 + s^{-1}\mathbf{u}_2 \otimes \mathbf{u}_2 + \mathbf{u}_3 \otimes \mathbf{u}_3)[t^{-1/2}(\mathbf{u}_1 \otimes \mathbf{u}_1 + \mathbf{u}_2 \otimes \mathbf{u}_2) \\ &\quad + t\mathbf{u}_3 \otimes \mathbf{u}_3],\end{aligned} \tag{8.31}$$

where $\lambda_i(> 0)$ are the principal stretches, $\{\mathbf{u}_i\}$ are the orthonormal principal axes of $\mathbf{U}$ and the factors correspond to a pure equi-triaxial stretch of amount $\lambda(> 0)$, a pure shear of amount $s(> 0)$, and an isochoric uniaxial extension of amount $t(> 0)$ with accompanying lateral contraction. These are coaxial and so may be composed in any order. [*Hint*: the problem is solved if you can establish an invertible relation between the $\{\lambda_i\}$ and $\{\lambda, s, t\}$. This would imply that the two expressions above for $\mathbf{U}$ are equivalent.] Show that the pure shear factor may be identified as the spectral decomposition of a simple-shear deformation on a fixed set of axes.

(b) In two dimensions, show that $\mathbf{F}$ may, without loss of generality, be decomposed in the form (8.30), where

$$\mathbf{U} = (\lambda \mathbf{I})(s\mathbf{u}_1 \otimes \mathbf{u}_1 + s^{-1}\mathbf{u}_2 \otimes \mathbf{u}_2) \tag{8.32}$$

is the composition of an areal dilation of amount λ and a pure shear of amount $s(> 0)$.

2. In two dimensions, use the spectral decomposition of $\mathbf{U}$ to derive

$$\mathbf{U} = I^{-1}(J\mathbf{I} + \mathbf{C}), \quad \text{where} \quad \mathbf{C} = \mathbf{F}^t\mathbf{F} = \mathbf{U}^2, \tag{8.33}$$

and, thus, obtain I directly in terms of the invariants of $\mathbf{C}$. Use this to obtain an explicit formula for $\mathbf{U}^{-1}$, and use it to confirm that $I\mathbf{R} = \mathbf{F} + \mathbf{F}^*$.

8.3 Radial expansion/compaction

Henceforth, we drop the subscript $(\cdot)_{\parallel}$ and consider deformations of the form

$$\mathbf{x} = R\mathbf{e}_r(\theta), \quad \mathbf{y} = r(R)\mathbf{e}_r(\theta). \tag{8.34}$$

We derive

$$\mathbf{F} = r'\mathbf{e}_r \otimes \mathbf{e}_r + (r/R)\mathbf{e}_\theta \otimes \mathbf{e}_\theta, \quad \mathbf{F}^* = (r/R)\mathbf{e}_r \otimes \mathbf{e}_r + r'\mathbf{e}_\theta \otimes \mathbf{e}_\theta \tag{8.35}$$

and

$$\mathbf{F} + \mathbf{F}^* = I\mathbf{I}, \quad \text{where} \quad I = R^{-1}(Rr)'. \tag{8.36}$$

Furthermore,

$$J = (r/R)r' \tag{8.37}$$

and the requirement $J > 0$ implies that $r(R)$ is an increasing function: $r' > 0$. Accordingly, in this case we have

$$\mathbf{P} = w_I \mathbf{I} + w_J \mathbf{F}^*. \tag{8.38}$$

Problem

Show that equilibrium without body force for is equivalent to the ordinary differential equation (compare with eqn (8.7))

$$(w_I)' + (r/R)(w_J)' = 0. \tag{8.39}$$

For uniform materials the trivial solution is

$$r(R) = \lambda_A R, \tag{8.40}$$

where $\lambda_A = r(A)/A$ and $r(A)$ is the (assigned) radius after deformation of the disc of initial radius A. To find a more interesting, yet tractable, alternative, consider again the special class of Varga-type materials

$$w = 2\mu[I + F(J)]. \tag{8.41}$$

Before proceeding, consider the response of such a material to a uniform equi-biaxial stretch, in which $\lambda_1 = \lambda_2 = J^{1/2}$, as exemplified by the trivial solution. In this mode of deformation the strain energy reduces to

$$w = 2\mu[2J^{1/2} + F(J)]. \tag{8.42}$$

The Piola stress components are $\partial\omega/\partial\lambda_1 = \partial\omega/\partial\lambda_2 = 2\mu P(J)$, where

$$P(J) = 1 + J^{1/2}F'(J). \tag{8.43}$$

We would expect, on physical grounds, that $P \to \pm\infty$ as $J \to \infty, 0$, respectively, and hence that $F'(J) \to -\infty$ as $J \to 0$ and $F'(J) > 0$ at large values of J. If, in addition, the energy and stress vanish in the undeformed state, then $F(1) = -2$ and $F'(1) = -1$. A function with all these properties, similar to that sketched in Figure 8.1, is depicted in Figure 8.3. This has an isolated minimum at some $J_0 > 1$.

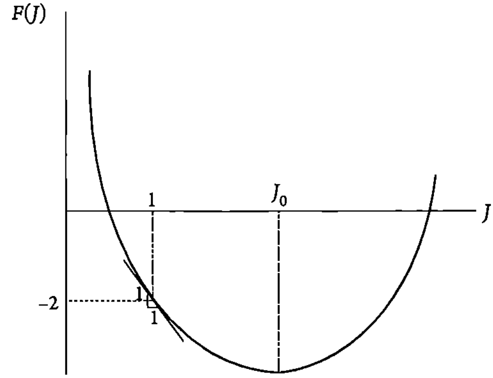

Figure 8.3 Constitutive response of a Varga-type material capable of supporting plane-strain cavitation

Problem

(a) Use this material to derive the general solution

$$r(R)^2 = J(R^2 - A^2) + \lambda_A^2 A^2. \tag{8.44}$$

Thus, $J = \lambda_A^2$ for the trivial solution. Show that a cavity forms if $J = J_0$ and $\lambda_A > J_0^{1/2}$, with radius $a = A(\lambda_A^2 - J_0)^{1/2}$. Plot this as a function of λ_A and show that it branches off the trivial solution at $\lambda_A = J_0^{1/2}$.

(b) Carry out an energy comparison and show that the cavitated equilibrium deformation minimizes the energy in the class of purely radial deformations, provided that $\lambda_A > J_0^{1/2}$.

(c) Plot the Piola traction at the outer edge of the disc as a function of λ_A and show that it increases without bound for the trivial solution, but saturates at a fixed value in the cavitated solution if $\lambda_A \geq J_0^{1/2}$.

Also of interest are the so-called *harmonic materials* defined by

$$w(I,J) = 2\mu[F(I) - J] \tag{8.45}$$

for some function F. These have the remarkable property that they yield explicit solutions to the general plane-strain equilibrium problem in terms of analytic functions of the complex variable $x_1 + ix_2$. However, they yield somewhat unrealistic predictions in deformations that induce severe compression. This is borne out by eqn (8.45), which furnishes the questionable prediction that a degenerate deformation with $J \to 0$ can be attained at a finite value of the energy. For this reason, the harmonic material is useful mainly in problems involving small-to-moderate strains with possibly finite rotations.

Problems

1. Show that all purely radial equilibrium deformations of harmonic materials are of the form

$$r(R) = IR/2 + C/R, \tag{8.46}$$

where I and C are constants. Show that cavitation is not possible in a harmonic material.

2. Consider a spherical shell of uniform, isotropic material, occupying the annular region $A \leq R \leq B$. Solve the equilibrium problem (no body force) in the class of radial deformations $\mathbf{x} \to \mathbf{y} = \lambda(R)\mathbf{x}$, where $\lambda(R) = r(R)/R$ and $R = |\mathbf{x}|$. Consider the following cases:

   (a) The material is compressible with strain energy given by $w/2\mu = f(i_1) - i_3$, where $\mu(> 0)$ is a material constant. Assume $f''(i_1) > 0$. State restrictions on f ensuring that the energy and stress vanish in the reference configuration. Assume the surface $R = B$ to be traction free and the surface $R = A$ to be subjected to pressure P.

   (b) The material is compressible with strain energy given by $w/2\mu = i_1 + g(i_3)$. Assume $g''(i_3) > 0$. State restrictions on g ensuring that the energy and stress vanish in the reference configuration. Same loading conditions as in (a).

   (c) Show how the addition of a term linear in i_2 to the strain–energy function affects the analyses of problems (a) and (b).

3. Recall that for plane strain of compressible isotropic materials the Piola stress may be written in the convenient form

$$\mathbf{P} = I^{-1}w_I(\mathbf{F} + \mathbf{F}^*) + w_J\mathbf{F}^*; \quad I = \lambda_1 + \lambda_2, \quad J = \lambda_1\lambda_2, \tag{8.47}$$

wherein all tensors are two-dimensional. Consider two-dimensional deformations $\mathbf{x} \to \mathbf{y}$ defined by

$$\mathbf{x} = R\mathbf{u}(\Theta), \quad \mathbf{y} = r(R)\mathbf{u}(\theta), \tag{8.48}$$

where $\mathbf{u} = \mathbf{e}_r$ and $\theta = \Theta + \Gamma(R)$. This combines radial expansion/contraction with azimuthal shear. It simplifies matters to write

$$\mathbf{y} = u(R)\mathbf{u}(\Theta) + v(R)\mathbf{v}(\Theta), \tag{8.49}$$

where $\mathbf{v}(x) = \mathbf{k} \times \mathbf{u}(x)$ and

$$u(R) = r(R)\cos\Gamma(R), \quad v(R) = r(R)\sin\Gamma(R). \tag{8.50}$$

Find a pair of coupled ODEs for $u(R)$ and $v(R)$. Solve them for the special case of harmonic materials with strain energies of the form $w/2\mu = F(I) - J$. Assume $F''(I) > 0$. This furnishes a good model if the strains are moderate while the rotations are large. Consider the BCs

$$r(B) = B, \quad r(A) = \lambda A; \quad \Gamma(B) = 0, \quad \Gamma(A) = \tau. \tag{8.51}$$

Is there a limit, according to this model, on the amount of rotation τ for prescribed $\lambda > 0$? Explain.

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9

Material stability, strong ellipticity and smoothness of equilibria

9.1 Small motions superposed on finitely deformed equilibrium states

Consider a small amplitude wave propagating through the elastic material. Suppose the material has been predeformed to some equilibrium state, with position field $\chi_\epsilon(\mathbf{x})$, prior to the passage of the wave. The wave causes the material point $\mathbf{x}$ to undergo a further displacement to position $\bar{\chi}(\mathbf{x}, t)$, say. Supposing the material to be incompressible and using obvious notation, the Piola stresses in these configurations are

$$\mathbf{P}_\epsilon = W_{\mathbf{F}}(\mathbf{F}_\epsilon; \mathbf{x}) - p_\epsilon \mathbf{F}_\epsilon^* \quad \text{and} \quad \bar{\mathbf{P}} = W_{\mathbf{F}}(\bar{\mathbf{F}}; \mathbf{x}) - \bar{p}\bar{\mathbf{F}}^*. \tag{9.1}$$

Here, we suppose the displacement from χ_ϵ to $\bar{\chi}$ to be small in the sense that

$$\bar{\chi}(\mathbf{x}, t; \epsilon) = \chi_\epsilon(\mathbf{x}) + \epsilon\chi'(\mathbf{x}, t) + o(\epsilon), \tag{9.2}$$

with $|\epsilon| \ll 1$, uniformly in $\mathbf{x}$ and t. We assume a concomitant change in the constraint pressure; i.e.,

$$\bar{p}(\mathbf{x}, t; \epsilon) = p_\epsilon(\mathbf{x}) + \epsilon p'(\mathbf{x}, t) + o(\epsilon), \tag{9.3}$$

and we seek a system valid to linear order in ϵ for the perturbation fields χ' and p'. We have made excessive use of the notation $(\cdot)'$, relying on the context to convey the intended meaning; here, derivatives with respect to ϵ, evaluated at $\epsilon = 0$.

We may use eqns (9.1)–(9.3) to deduce that

$$\bar{\mathbf{P}}(\mathbf{x}, t; \epsilon) = \mathbf{P}_\epsilon(\mathbf{x}) + \epsilon\mathbf{P}'(\mathbf{x}, t) + o(\epsilon), \tag{9.4}$$

Finite Elasticity Theory. David J. Steigmann.

where

$$\mathbf{P}' = \mathcal{M}[\mathbf{F}'] - p'\mathbf{F}_e^* - p_e\mathbf{F}_{\mathbf{F}}^*[\mathbf{F}'], \tag{9.5}$$

wherein the derivatives

$$\mathcal{M} = W_{\mathbf{FF}} \tag{9.6}$$

and $\mathbf{F}_{\mathbf{F}}^*$ are evaluated at $\mathbf{F}_e$. In terms of components,

$$\mathcal{M}_{iAjB} = \partial^2 W/\partial F_{iA}\partial F_{jB}. \tag{9.7}$$

Problem

Show that $\mathcal{M}$ possesses major symmetry, i.e., $\mathcal{M} = \mathcal{M}^t$, where, for 4th order tensors, the transpose is defined by $\mathbf{A} \cdot \mathcal{M}^t[\mathbf{B}] = \mathbf{B} \cdot \mathcal{M}[\mathbf{A}]$. Thus, $\mathcal{M}_{iAjB} = \mathcal{M}_{jBiA}$.

Assuming zero body force for simplicity, we now substitute eqn (9.4) into the equation of motion

$$Div\bar{\mathbf{P}} = \rho_\kappa \ddot{\bar{\chi}}, \tag{9.8}$$

divide the result by ϵ, and let $\epsilon \to 0$ to arrive at the linear differential equation

$$Div\mathbf{P}' = \rho_\kappa \ddot{\mathbf{u}}, \tag{9.9}$$

where

$$\mathbf{u} = \chi' \tag{9.10}$$

is the linear approximation to the small displacement, and

$$\mathbf{F}' = \nabla\mathbf{u}. \tag{9.11}$$

The constraint of incompressibility imposes a restriction on $\nabla\mathbf{u}$. To see this, we write

$$\bar{J} = J_e + \epsilon J' + o(\epsilon), \tag{9.12}$$

with $\bar{J} = J_e = 1$, divide by ϵ, and let $\epsilon \to 0$ to obtain $J' = 0$. However, $J' = J_{\mathbf{F}} \cdot \mathbf{F}'$, yielding the restriction

$$\mathbf{F}_e^* \cdot \nabla\mathbf{u} = 0. \tag{9.13}$$

This and eqn (9.9) provide the system to determine $\mathbf{u}$ and p', subject to appropriate boundary and initial conditions.

If the underlying equilibrium deformation $\chi_e(\mathbf{x})$ is homogeneous and if the material is uniform—this situation being the simplest—then $\mathbf{F}_e$ and p_e are uniform. In particular, $\mathcal{M}$ is then uniform and equation (9.9) simplifies to

$$\rho_\kappa \ddot{u}_i = (\mathcal{M}_{iAjB} - p_e \partial F^*_{iA}/\partial F_{jB})u_{j,BA} - F^*_{iA}p'_{,A} \tag{9.14}$$

Problem

Show that $J\partial F^*_{iA}/\partial F_{jB} = F^*_{iA}F^*_{jB} - F^*_{jA}F^*_{iB}$ and, hence, that $(\partial F^*_{iA}/\partial F_{jB})u_{j,BA} = 0$. The latter is the linearized form of the Piola identity $F^*_{iA,A} = 0$.

Thus,

$$\rho_\kappa \ddot{u}_i = \mathcal{M}_{iAjB}u_{j,BA} - F^*_{iA}p'_{,A}, \quad \text{with} \quad F^*_{iA}u_{i,A} = 0. \tag{9.15}$$

The compressible case is recovered by omitting the Lagrange multiplier p' and suppressing the second equation.

Consider a *plane harmonic* wave of the form

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a}\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)], \quad p'(\mathbf{x},t) = q\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)], \tag{9.16}$$

wherein the constant vectors $\mathbf{a}$ and $\mathbf{k}$ are the polarization and wave vectors, respectively. The constant q is the amplitude of the perturbed constraint pressure; the constant ω is the frequency, and i is the complex unit ($i^2 = -1$). We show that this furnishes a solution to eqn (9.15). Naturally, these simple functions are not able to satisfy initial or boundary conditions, and so we suppose that the wave has been propagating for some time in an unbounded medium. Equivalently, attention is confined to an interval of time prior to impingement of the wave on the boundaries of the body.

A convenient alternative representation of the waveform is obtained by introducing the wave number $k = |\mathbf{k}|$ and wavespeed $c = \omega/k$. Then,

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a}\exp[ik(\mathbf{N}\cdot\mathbf{x} - ct)] \quad \text{and} \quad p'(\mathbf{x},t) = q\exp[ik(\mathbf{N}\cdot\mathbf{x} - ct)], \tag{9.17}$$

where $\mathbf{N} = k^{-1}\mathbf{k}$. These make clear the fact that the wave form is preserved on the plane defined by

$$\mathbf{N}\cdot\mathbf{x} = ct + d, \tag{9.18}$$

with unit normal $\mathbf{N}$. Here, d is the perpendicular distance from the plane to the origin at time zero. Thus, the distance from the plane to the origin changes with velocity c.

Evidently, this perturbation of the underlying equilibrium solution remains bounded in amplitude provided that c or ω is a real number. This case is referred to as *material* stability, to highlight the fact that no boundary or initial conditions are involved and, hence, that the stability or otherwise of the underlying solution depends entirely on the properties of the material *per se*. Of course, it remains to verify that eqn (9.17) furnishes a solution to eqn (9.15). Simple calculations give

$$u_{j,A} = iu_j k_A, \quad u_{j,AB} = -u_j k_A k_B, \quad \ddot{u}_j = -\omega^2 u_j \quad \text{and} \quad p'_{,A} = ip' k_A \tag{9.19}$$

and eqn (9.13) yields the restriction

$$\mathbf{u} \cdot \mathbf{F}_e^* \mathbf{k} = 0, \tag{9.20}$$

which, by virtue of Nanson's formula, requires that the displacement be polarized in the image of the plane defining the plane wave in the deformed equilibrium configuration. Using these results, eqn $(9.15)_1$ is reduced to the algebraic equation

$$\mathbf{A}(\mathbf{F}_e, \mathbf{k})\mathbf{u} + ip'\mathbf{F}_e^*\mathbf{k} = \rho_\kappa \omega^2 \mathbf{u}, \tag{9.21}$$

where $\mathbf{A}(\mathbf{F}, \mathbf{k})$ is the so-called *acoustic* tensor, having components

$$A_{ij} = \mathcal{M}_{iAjB}(\mathbf{F}) k_A k_B. \tag{9.22}$$

Problem

Use the major symmetry of $\mathcal{M}$ to demonstrate that $\mathbf{A}$ is symmetric.

From eqn (9.21) it follows that

$$\mathbf{u} \cdot \mathbf{A}(\mathbf{F}_e, \mathbf{k})\mathbf{u} = \rho_\kappa \omega^2 |\mathbf{u}|^2, \tag{9.23}$$

and the remaining content of eqn (9.21) is

$$\mathbf{F}_e^*\mathbf{k} \cdot \mathbf{A}(\mathbf{F}_e, \mathbf{k})\mathbf{u} + ip' |\mathbf{F}_e^*\mathbf{k}|^2 = 0, \tag{9.24}$$

which determines p' in terms of $\mathbf{u}$. Our procedure can yield complex values because of the assumption eqn (9.16). To rectify this, we can qualify the latter by taking real or imaginary parts a priori. We conclude that p' is bounded if, and only if, $\mathbf{u}$ is bounded. The first result implies that $\omega^2 > 0$; hence, $\omega \in \mathbb{R}$ and material stability, whenever $\mathbf{A}$ is positive definite. Whether or not this is the case evidently depends only on $\mathbf{F}_e$ and the strain–energy function.

In the case of a compressible material the constraint eqn (9.13) is not relevant and the foregoing results remain valid with the Lagrange multiplier suppressed. Equation (9.21) is replaced by the eigen-problem

$$\mathbf{A}(\mathbf{F}_e, \mathbf{k})\mathbf{u} = \rho_\kappa \omega^2 \mathbf{u}. \tag{9.25}$$

Because $\mathbf{A}$ is symmetric, it has three real eigenvalues ω^2 and three mutually orthogonal eigenvectors—the polarization vectors. Material stability obtains if, and only if, all eigenvalues are positive and, hence, if and only if $\mathbf{A}$ is positive definite.

Problems

1. According to the foregoing analysis the equilibrium state is stable with respect to perturbations of the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{a} \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \tag{9.26}$$

if the associated *acoustic tensor* is positive definite. Show that an unstable solution exists if the acoustic tensor has a negative eigenvalue. Furthermore, show that an unstable solution of the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{a}t \exp(i\mathbf{k} \cdot \mathbf{x}) \tag{9.27}$$

exists when the acoustic tensor is positive *semi*-definite. Conclude that strict positive definiteness of the acoustic tensor is a necessary condition for stability.

2. Consider the propagation of infinitesimal waves superposed on a static finite deformation (in equilibrium without body force) of a homogeneous *incompressible* elastic solid. Suppose the underlying finite deformation is a homogeneous triaxial stretch with deformation gradient $\mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{E}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{E}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{E}_3$, where λ_i are positive constants and $\{\mathbf{e}_i\} = \{\mathbf{E}_A\}$. Consider plane harmonic waves superposed on the static solution. Obtain an expression for the acoustic tensor using the *neo-Hookean* strain-energy function and analyze the associated eigenvalue problem for the wavespeeds. Are there conditions under which the wavespeeds can be imaginary numbers?

9.2 Smoothness of equilibria

Suppose a given equilibrium deformation $\chi(\mathbf{x})$ is C^2, i.e., twice differentiable in the sense that χ_i, F_{iA} and $F_{iA,B}$ are continuous functions of $\mathbf{x}$. It satisfies

$$P_{iA,A} + \rho_\kappa b_i = 0 \tag{9.28}$$

everywhere in the body, with

$$P_{iA} = \partial W / \partial F_{iA}. \tag{9.29}$$

We consider only the unconstrained case for now.

The chain rule furnishes

$$P_{iA,A} = \mathcal{M}_{iAjB}F_{jB,A} + R_i, \tag{9.30}$$

where

$$R_i = \partial^2 W/\partial x_A \partial F_{iA}, \tag{9.31}$$

and eqn (9.28) reduces to

$$\mathcal{M}_{iAjB}F_{jB,A} + \rho_\kappa b_i + R_i = 0. \tag{9.32}$$

Suppose, instead, that $\chi(\mathbf{x})$ is C^1 and piecewise C^2, i.e., χ_i and F_{iA} are continuous, but $F_{iA,B}$ may jump across one or more surfaces in the body. We want to derive conditions that allow for this possibility, but first we need some preliminary discussion.

Consider a patch of surface, described in parametric form by the position field $\mathring{\mathbf{x}}(u_1, u_2)$, and let $\mathbf{N}(u_1, u_2)$ be a unit-normal field on this patch. The surface divides the reference configuration into two parts, denoted by + and –. A point off the surface may be located by specifying the value of ς in the normal-coordinate parametrization (Figure 9.1)

$$\mathbf{x}(u_1, u_2, \varsigma) = \mathring{\mathbf{x}}(u_1, u_2) + \varsigma\mathbf{N}(u_1, u_2) \tag{9.33}$$

of the surrounding 3-space. It is easy to demonstrate that the relationship between the coordinates $\{u_1, u_2, \varsigma\}$ and $\mathbf{x}$ is invertible in any sufficiently small three-dimensional neighborhood of a point on the surface. In particular, there is a one-to-one relationship among the Cartesian coordinates x_A and $\{u_\alpha, \varsigma\}$ in this neighborhood; we use Greek subscripts, ranging over $\{1, 2\}$, to identify surface coordinates.

Furthermore, we assume these relations to be as smooth as required by the analysis. Confining attention to such a neighborhood, we are then justified in writing

$$d\varsigma = \nabla\varsigma \cdot d\mathbf{x}, \tag{9.34}$$

where

$$d\mathbf{x} = \mathring{\mathbf{x}}_{,\alpha}du_\alpha + \mathbf{N}d\varsigma + \varsigma d\mathbf{N}. \tag{9.35}$$

Because $\mathring{\mathbf{x}}_{,\alpha}$ and $d\mathbf{N}$ are tangential to the surface, it then follows that

$$d\varsigma = \mathbf{N} \cdot d\mathbf{x}, \tag{9.36}$$

and comparison with eqn (9.34) yields

$$\mathbf{N} = \nabla\varsigma. \tag{9.37}$$

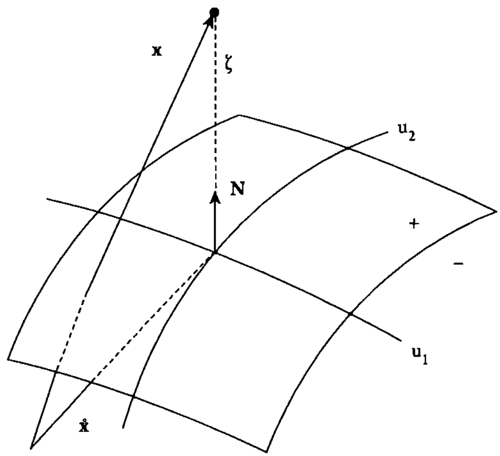

Figure 9.1 Normal-coordinate parametrization of three-space in the vicinity of a surface

For any function f of position, and hence of the coordinates, we define the upper and lower surface limits

$$f_+(u_1, u_2) = \lim_{\varsigma \to 0^+} f(u_1, u_2, \varsigma) \quad \text{and} \quad f_-(u_1, u_2) = \lim_{\varsigma \to 0^-} f(u_1, u_2, \varsigma), \tag{9.38}$$

and the associated discontinuity

$$[f] = f_+(u_1, u_2) - f_-(u_1, u_2). \tag{9.39}$$

If the deformation $\chi(\mathbf{x})$ is continuous across the surface, i.e., if the material does not fracture, then the χ_i are continuous, i.e., $\chi_i^+(u_1, u_2) = \chi_i^-(u_1, u_2)$. Assuming sufficient smoothness of these limits *on* the surface, we can differentiate and conclude that

$$[\chi_{i,\alpha}] = [\chi_{i,\alpha\beta}] = 0, \quad \text{etc.} \tag{9.40}$$

However, there is no reason to conclude that there is any relationship between the limits

$$\chi^+_{i,\varsigma}(u_1, u_2) = \lim_{\varsigma \to 0^+} (\partial \chi_i / \partial \varsigma) \quad \text{and} \quad \chi^-_{i,\varsigma}(u_1, u_2) = \lim_{\varsigma \to 0^-} (\partial \chi_i / \partial \varsigma). \tag{9.41}$$

In a small neighborhood of the surface, we may apply the chain rule to derive

$$F_{jB} = \chi_{j,\alpha} u_{\alpha,B} + \chi_{j,\varsigma} \varsigma_{,B}. \tag{9.42}$$

The jump across the surface is then found to be

$$[F_{jB}] = a_j N_B, \quad \text{where} \quad a_j = [\chi_{j,\varsigma}]. \tag{9.43}$$

The discontinuity in the deformation gradient is thus necessarily of the form

$$[\mathbf{F}] = \mathbf{a} \otimes \mathbf{N} \tag{9.44}$$

for some vector $\mathbf{a}$. Tensors of this type are said to be *rank-1*. Here, as in the theory of matrices, the *rank* of a tensor $\mathbf{A}$, say, is equal to the dimension of its image space: $Rank\mathbf{A} = \dim\{\mathbf{Av}\}$, where $\mathbf{v}$ is any vector. In the present example the image space is the one-dimensional space spanned by $\mathbf{a}$.

Suppose now that $\chi(\mathbf{x})$ is C^1. Then $[\mathbf{F}]$ vanishes and so $\mathbf{a}$ vanishes: $[\chi_{i,\varsigma}] = 0$. We have $\chi^+_{i,\varsigma}(u_1, u_2) = \chi^-_{i,\varsigma}(u_1, u_2)$, and assuming these surfacial limits to be smooth, it follows that $[\chi_{i,\varsigma\alpha}] = 0$. Proceeding from eqn (9.42), we compute

$$\begin{aligned} F_{jB,A} = {} & \chi_{j,\alpha} u_{\alpha,BA} + \chi_{j,\varsigma} \varsigma_{,BA} + \chi_{j,\alpha\beta} u_{\alpha,B} u_{\beta,A} \\ & + \chi_{j,\varsigma\alpha} u_{\alpha,A} \varsigma_{,B} + \chi_{j,\alpha\varsigma} u_{\alpha,B} \varsigma_{,A} + \chi_{j,\varsigma\varsigma} \varsigma_{,A} \varsigma_{,B}. \end{aligned} \tag{9.45}$$

Taking jumps then yields

$$[F_{jB,A}] = a_j N_B N_A, \quad \text{where} \quad a_j = [\chi_{j,\varsigma\varsigma}]. \tag{9.46}$$

Next, we take limits of eqn (9.32) as the surface is approached from above and below, obtaining

$$\mathcal{M}_{iAjB} F^{\pm}_{jB,A} + \ldots = 0, \tag{9.47}$$

wherein the missing terms are continuous across the surface. Subtracting the two equations and invoking eqn (9.46), we find that

$$\mathcal{M}_{iAjB} N_A N_B a_j = 0, \tag{9.48}$$

or

$$\mathbf{A}(\mathbf{F}, \mathbf{N})\mathbf{a} = \mathbf{0}, \tag{9.49}$$

where $\mathbf{A}(\mathbf{F}, \mathbf{N})$ is the acoustic tensor based on $\mathbf{N}$. It follows that a discontinuity is possible, i.e., $\mathbf{a} \neq \mathbf{0}$, if and only if $\det \mathbf{A}(\mathbf{F}, \mathbf{N}) = 0$. This is an equation for the local orientation $\mathbf{N}$ of the discontinuity surface. On the other hand, if the strain–energy function is such that $\mathbf{A}(\mathbf{F}, \mathbf{N})$ is non-singular for any deformation, i.e., if the equations of equilibrium are always of *elliptic* type, then eqn (9.49) requires that the discontinuity vanish and the underlying deformation is C^2. It is possible to continue in a recursive manner to show that if an equilibrium deformation is piecewise C^n, then it is, in fact, C^n for any n, provided that the acoustic tensor is non-singular, granted sufficient regularity of the function $W(\mathbf{F})$.

Problem

Verify this claim. *Hint*: the ς—derivatives of the deformation are the only ones having potential discontinuities.

Note that material stability, which is tantamount to the positive definiteness of the acoustic tensor and which confers its nonsingular character, is enough to ensure that the smoothness conditions to which we have referred are fulfilled. However, this falls short of proving that equilibria are arbitrarily smooth in the presence of strong ellipticity. For, there is no known proof of the piecewise C^2 continuity that was presumed at the outset, although partial results of this kind are known for a restricted class of boundary data (see the paper by Healey and Rosakis, 1997).

9.3 Incompressibility

We have seen that if the material is incompressible then the stress is given by

$$P_{iA} = \partial W/\partial F_{iA} - pF^{*}_{iA}. \tag{9.50}$$

Suppose that F_{iA} and p are continuous functions of $\mathbf{x}$, but that their gradients may be discontinuous across some surface. Then, as before,

$$[F_{jB,A}] = a_j N_B N_A \quad \text{and} \quad [p_{,A}] = qN_A, \tag{9.51}$$

for some a_j and q. The second of these is derived as eqn (9.44) was derived, on replacing χ_i by p. On either side of the discontinuity surface, eqn (9.28) applies and yields

$$\mathcal{M}_{iAjB}F_{jB,A} - F^{*}_{iA}p_{,A} + \ldots = 0, \tag{9.52}$$

wherein the missing terms are continuous. Taking limits from above and below this surface, and subtracting the resulting equations, as before, we arrive at

$$\mathbf{A}(\mathbf{F},\mathbf{N})\mathbf{a} = q\mathbf{F}^{*}\mathbf{N}. \tag{9.53}$$

Here, however, $\mathbf{a}$ is subject to the restriction $J(\mathbf{F}(\mathbf{x})) = 1$, identically, at all points removed from the discontinuity surface. This implies that ∇J vanishes identically in the body, minus the surface. Using the chain rule, this is found to be equivalent to

$$F^{*}_{iA}F_{iA,B} = 0. \tag{9.54}$$

Taking the jump, we arrive at

$$\mathbf{a} \cdot \mathbf{F}^{*}\mathbf{N} = 0. \tag{9.55}$$

Accordingly, eqn (9.53) requires that $\mathbf{a} \cdot \mathbf{A}(\mathbf{F}, \mathbf{N})\mathbf{a}$ vanishes if a discontinuity is to exist. It follows that if the strong ellipticity—or material stability—condition is satisfied, then the only resolution is $\mathbf{a} = \mathbf{0}$; then, eqn (9.53) yields $q = 0$, and there is no discontinuity. Proceeding by recursion, it is possible to show that both the deformation and pressure fields are arbitrarily smooth, granted the degree of continuity assumed at the outset.

Problem

Prove this.

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10

Membrane theory

A membrane is a thin sheet of thickness h, which is much smaller than any spanwise dimension of the sheet, such as its overall diameter or the diameter of an interior hole. Membranes provide a particularly useful setting for the empirical testing of formulations for the strain-energy function. Their relatively easy deformability affords empirical access to large regions of strain space. In this respect they are similar to rubber bands in furnishing archetypal examples through which elasticity can be understood, both qualitatively and quantitatively. Indeed, the empirical work of Treloar (1975) on rubber elasticity—arguably the most important collection of work of its kind—was conducted on thin membranes.

10.1 General theory

Our intention, here, is to exploit the thinness of the membrane to derive an approximate two-dimensional theory that captures the most important aspects of the behavior of thin sheets. We concentrate on equilibria, although extensions to accommodate dynamics are straightforward. To this end, let l be the next smallest length scale in the problem at hand, such as a spanwise dimension or the length scale for the spatial variation of a distribution of load. We suppose that $h/l \ll 1$, and proceed to derive the leading-order two-dimensional approximation to the three-dimensional equations. This leading-order approximate model is what we mean by *membrane theory*. To highlight its important features, we consider the simplest case in which the reference configuration κ is a thin prismatic plate-like region of three-space. In this simplest case, we can decompose the reference configuration into the Cartesian product of a *midplane* Ω and a through-thickness *fiber* C: $\kappa = \Omega \times C$ (Figure 10.1). Correspondingly, we write position in κ as

$$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{u}, \varsigma), \quad \text{where} \quad \hat{\mathbf{x}}(u, \varsigma) = \mathbf{u} + \varsigma\mathbf{k}. \tag{10.1}$$

Here, $\mathbf{u}$ is (two-dimensional) position on Ω, $\varsigma \in C$ is a linear through-thickness coordinate and $\mathbf{k}$ is the (fixed) unit normal to Ω. Thus, C may be identified with the interval $[-h/2, h/2]$ containing ς. We suppose that all length scales have been non-dimensionalized by l; equivalently, we adopt l as the unit of length ($l = 1$) and assume that $h \ll 1$.

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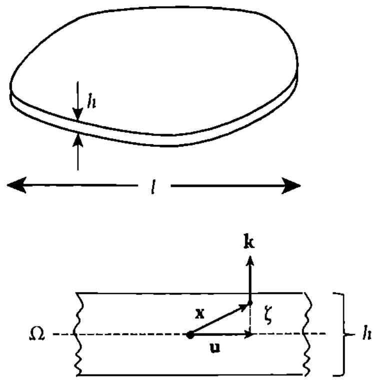

Figure 10.1 Reference configuration of a thin sheet

In a deformation $\chi(\mathbf{x})$ of the body, the plane Ω is carried to a surface ω, described parametrically by

$$\mathbf{r}(\mathbf{u}) = \chi(\hat{\mathbf{x}}(\mathbf{u}, 0)). \tag{10.2}$$

As we shall see, the determination of this function is the main objective of membrane theory. As usual, the problem to be solved is

$$Div\mathbf{P} = \mathbf{0} \tag{10.3}$$

in κ, subject to some set of boundary conditions. We exclude body forces for the sake of brevity and convenience; their inclusion presents little difficulty. Equivalently,

$$0 = P_{iA,A} = P_{i\alpha,\alpha} + P'_{i3}, \tag{10.4}$$

where Greek indices range over $\{1, 2\}$ and the prime stands for $\partial(\cdot)/\partial\varsigma$. In other words, we have identified ς with x_3. We thus identify x_α with u_α, the Cartesian coordinates of $\mathbf{u}$; and $\mathbf{E}_3$ with $\mathbf{k}$.

This suggests the decomposition

$$\mathbf{P} = \mathbf{P1} + \mathbf{Pk} \otimes \mathbf{k}, \tag{10.5}$$

where

$$\mathbf{1} = \mathbf{I} - \mathbf{k} \otimes \mathbf{k} \tag{10.6}$$

is the projection onto the plane Ω. Here, we have used this to expand the identity $\mathbf{P} = \mathbf{PI}$. Furthermore, $\mathbf{1} = \mathbf{E}_\alpha \otimes \mathbf{E}_\alpha$ and so

$$\mathbf{P1} = P_{i\alpha}\mathbf{e}_i \otimes \mathbf{E}_\alpha. \tag{10.7}$$

Equation (10.4) is then seen to be the component form of

$$Div_{\|}(\mathbf{P1}) + \mathbf{P}'\mathbf{k} = \mathbf{0}, \tag{10.8}$$

where $Div_{\|}(\cdot)$ is the two-dimensional divergence with respect to position $\mathbf{u}$. This is just another way of writing eqn (10.3); accordingly, it holds at all points of κ and, hence, on the plane Ω in particular, i.e., at $\varsigma = 0$, where it reduces to

$$Div_{\|}(\mathbf{P}_0\mathbf{1}) + \mathbf{P}_0'\mathbf{k} = \mathbf{0}. \tag{10.9}$$

The subscript $(\cdot)_0$ refers to function values on the plane, and $(\cdot)_0'$ refers to a ς - derivative, evaluated at $\varsigma = 0$.

We will focus attention on uniform incompressible materials, for which

$$\mathbf{P}_0 = W_{\mathbf{F}}(\mathbf{F}_0) - q_0\mathbf{F}_0^*. \tag{10.10}$$

We use q instead of p to denote the constraint pressure, for reasons that will become clear as we proceed. Let $\hat{\chi}(\mathbf{u}, \varsigma) = \chi(\hat{\mathbf{x}}(\mathbf{u}, \varsigma))$. Then,

$$\mathbf{F}d\mathbf{x} = d\chi = (\nabla\hat{\chi})d\mathbf{u} + \hat{\chi}'d\varsigma, \tag{10.11}$$

where, for the purposes of this chapter, ∇ is the (two-dimensional) gradient with respect to $\mathbf{u}$, whereas, from eqn (10.1),

$$d\mathbf{x} = d\mathbf{u} + \mathbf{k}d\varsigma. \tag{10.12}$$

As $d\mathbf{u} = \mathbf{1}d\mathbf{u}$, we have

$$\mathbf{F}d\mathbf{x} = \mathbf{F1}d\mathbf{u} + \mathbf{Fk}d\varsigma, \tag{10.13}$$

and comparison with (10.11), together with $\mathbf{F} = \mathbf{F1} + \mathbf{Fk} \otimes \mathbf{k}$, yields

$$\mathbf{F} = \nabla\hat{\chi} + \hat{\chi}' \otimes \mathbf{k}. \tag{10.14}$$

Evaluating this at the midplane, we obtain

$$\mathbf{F}_0 = \nabla\mathbf{r}(\mathbf{u}) + \mathbf{d}(\mathbf{u}) \otimes \mathbf{k}, \tag{10.15}$$

for use in eqn (10.10), where $\mathbf{d} = \hat{\chi}_0'$. Observe that $\mathbf{r}(\mathbf{u})$ and $\mathbf{d}(\mathbf{u})$ are independent vector fields on Ω. In the literature, $\mathbf{d}$ is often referred to as the *director* field. It represents the tangent to a material curve after deformation, evaluated at $\varsigma = 0$, and oriented perpendicular to Ω in κ. The first term on the right in eqn (10.15) has the representation

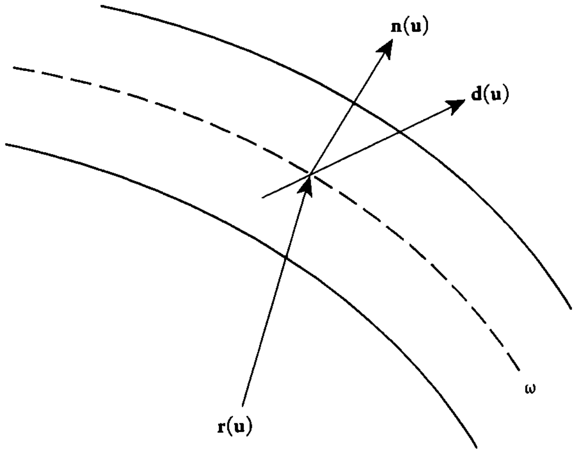

Figure 10.2 Position and director fields on the deformed surface

$$\nabla \mathbf{r} = \mathbf{r}_{,\alpha} \otimes \mathbf{E}_\alpha \tag{10.16}$$

in obvious notation. Because the $\mathbf{r}_{,\alpha}$ lie tangential to the deformed midsurface ω, this tensor maps Ω to the tangent plane to ω at the material point associated with $\mathbf{u}$ (see Figure 10.2).

The areal stretch α and orientation $\mathbf{n}$ of the material surface ω may be inferred from Nanson's formula. Thus,

$$\begin{aligned}\alpha\mathbf{n} &= \mathbf{F}_0^*\mathbf{k} = \mathbf{F}_0\mathbf{E}_1 \times \mathbf{F}_0\mathbf{E}_2 \\ &= (\nabla\mathbf{r})\mathbf{E}_1 \times (\nabla\mathbf{r})\mathbf{E}_2 = \mathbf{r}_{,1} \times \mathbf{r}_{,2},\end{aligned} \tag{10.17}$$

implying that

$$\alpha = |\mathbf{r}_{,1} \times \mathbf{r}_{,2}|. \tag{10.18}$$

Furthermore,

$$\begin{aligned}J_0 &= \det \mathbf{F}_0 = [\mathbf{F}_0\mathbf{E}_1, \mathbf{F}_0\mathbf{E}_2, \mathbf{F}_0\mathbf{E}_3] \\ &= \mathbf{F}_0\mathbf{E}_1 \times \mathbf{F}_0\mathbf{E}_2 \cdot \mathbf{F}_0\mathbf{k} = \mathbf{r}_{,1} \times \mathbf{r}_{,2} \cdot \mathbf{d},\end{aligned} \tag{10.19}$$

and so

$$J_0 = \alpha\mathbf{n} \cdot \mathbf{d}. \tag{10.20}$$

Accordingly, if the deformation is isochoric, as it must be for incompressible materials, then

$$\alpha\mathbf{n} \cdot \mathbf{d} = 1. \tag{10.21}$$

This yields the conclusion that

$$\mathbf{d} = \alpha^{-1}\mathbf{n} + (\nabla\mathbf{r})\mathbf{e}, \tag{10.22}$$

where $\mathbf{e}$ is a two-vector lying in Ω.

Thus far, we have merely recast the equations without invoking any approximations. We do so now, by estimating the lateral-traction boundary conditions. For example, if $\mathbf{p}^{\pm}$ are the tractions acting at the major surfaces $\varsigma = \pm h/2$ with unit normals $\mathbf{N} = \pm\mathbf{k}$, then $\mathbf{p}^{\pm} = \pm\mathbf{P}^{\pm}\mathbf{k}$. Accordingly, for small h, we have $\mathbf{p}^{\pm} = \pm\mathbf{P}_0\mathbf{k} + (h/2)\mathbf{P}_0'\mathbf{k} + o(h)$. The net-force density and the force-difference density on these surfaces are thus approximated by

$$\mathbf{p}^{+} + \mathbf{p}^{-} = h\mathbf{P}_0'\mathbf{k} + o(h) \quad \text{and} \quad \mathbf{p}^{+} - \mathbf{p}^{-} = 2\mathbf{P}_0\mathbf{k} + o(h). \tag{10.23}$$

It may be noted that the degree of differentiability required by these estimates is consistent with our earlier discussion about smoothness of equilibria in the presence of strong ellipticity.

Substituting the first estimate into the exact equation eqn (10.9), we derive

$$Div_{\|}(\mathbf{P}_0\mathbf{1}) + h^{-1}(\mathbf{p}^{+} + \mathbf{p}^{-}) + h^{-1}o(h) = \mathbf{0}. \tag{10.24}$$

An attempt to balance the terms reveals that $\mathbf{p}^{+} + \mathbf{p}^{-}$ can be of order h, at most (including the possibility that it vanishes) and, hence, that

$$\mathbf{p}^{+} + \mathbf{p}^{-} = h\mathbf{p} + o(h), \tag{10.25}$$

where $\mathbf{p}$ is a vector field of order unity. In the same way, eqn (10.23), part 2, indicates that

$$\mathbf{p}^{+} - \mathbf{p}^{-} = 2\mathbf{q} + o(1), \tag{10.26}$$

where $\mathbf{q}$ is likewise of order unity. Inserting these into eqns (10.24) and (10.23), part 2, and passing to the limit $h \to 0$, furnishes the leading-order differential-algebraic problem

$$Div_{\|}(\mathbf{P}_0\mathbf{1}) + \mathbf{p} = \mathbf{0}, \quad \mathbf{P}_0\mathbf{k} = \mathbf{q} \tag{10.27}$$

for the determination of the fields $\mathbf{r}(\mathbf{u})$ and $\mathbf{d}(\mathbf{u})$. In the case of incompressibility we have a system for the determination of $\{\mathbf{r}(\mathbf{u}), \mathbf{e}(\mathbf{u}), q_0(\mathbf{u})\}$.

10.2 Pressurized membranes

An important example is furnished by lateral pressure loading. Suppose the upper surface $\varsigma = h/2$ is traction free, while the lower surface $\varsigma = -h/2$ is loaded by a pressure of intensity p. Then, of course, $\mathbf{p}^{+} = \mathbf{0}$.

Problem

Show that $\mathbf{p}^- = p(\mathbf{F}^*)^-\mathbf{k}$.

Thus, $\mathbf{p}^- = p[\mathbf{F}_0^*\mathbf{k} + O(h)]$, yielding

$$p[\mathbf{F}_0^*\mathbf{k} + O(h)] = h\mathbf{p} + o(h) \quad \text{and} \quad -p[\mathbf{F}_0^*\mathbf{k} + O(h)] = 2\mathbf{q} + o(1). \tag{10.28}$$

These are reconciled by taking

$$p = hP + o(h),$$

with P of order unity. It follows that $\mathbf{p} = P\mathbf{F}_0^*\mathbf{k}$ and $\mathbf{q} = \mathbf{0}$, and the equations to be solved are

$$Div_{\|}(\mathbf{P}_0\mathbf{1}) + \alpha P\mathbf{n} = \mathbf{0} \quad \text{and} \quad \mathbf{P}_0\mathbf{k} = \mathbf{0}. \tag{10.29}$$

The pressure is seen to contribute a force that is distributed over the membrane surface, in the same way that a conventional body force is distributed over a body's volume. The second equation implies that the membrane is in a state of plane stress, at leading order. In the older literature, conditions of the latter type were typically imposed, rather than derived, as we have done. This is unnatural, however, and obscures the logical structure of the theory.

10.3 Uniqueness of the director

Observe that the plane-stress condition, or alternatively eqn (10.27), part 2, amounts, via eqns (10.15), (10.22), and (10.10), to an algebraic relationship among the entries of $\{\mathbf{r}(\mathbf{u}), \mathbf{e}(\mathbf{u}), q_0(\mathbf{u})\}$. In the next subsection—on isotropic materials—we will use it to evaluate $\mathbf{e}$ and q_0 in terms of the midplane deformation $\mathbf{r}(\mathbf{u})$. Before doing so, we would like to know whether or not such solutions are unique. We proceed to answer this question in the affirmative, with the proviso that the strong-ellipticity condition is satisfied.

First, observe that eqn (10.29), part 2, is equivalent to the statement $\mathbf{F}^t(W_{\mathbf{F}})\mathbf{k} = q\mathbf{k}$; here and, henceforth, we drop the subscript $(\cdot)_0$ for convenience. This, in turn, is equivalent to

$$\mathbf{0} = \mathbf{1}\mathbf{F}^t(W_{\mathbf{F}})\mathbf{k} = (\nabla\mathbf{r})^t(W_{\mathbf{F}})\mathbf{k} \quad \text{and} \quad q = \mathbf{k}\cdot\mathbf{F}^t(W_{\mathbf{F}})\mathbf{k} = \mathbf{d}\cdot(W_{\mathbf{F}})\mathbf{k}. \tag{10.30}$$

Next, let us fix $\mathbf{r}(\mathbf{u})$ in the function $W(\mathbf{F})$, where $\mathbf{F}$ is given by (10.15) and (10.22). This results in a function of $\mathbf{e}$, which we denote by $G(\mathbf{e})$. Consider a path $\mathbf{e}(u)$ in the space of two-vectors, and let $\sigma(u) = G(\mathbf{e}(u))$. This has the derivative

$$\dot{\sigma} = W_{\mathbf{F}}\cdot\dot{\mathbf{d}}\otimes\mathbf{k} = W_{\mathbf{F}}\cdot(\nabla\mathbf{r})\dot{\mathbf{e}}\otimes\mathbf{k} = (\nabla\mathbf{r})\dot{\mathbf{e}}\cdot(W_{\mathbf{F}})\mathbf{k} = \dot{\mathbf{e}}\cdot(\nabla\mathbf{r})^t(W_{\mathbf{F}})\mathbf{k}, \tag{10.31}$$

from which it follows that

$$G_{\mathbf{e}} = (\nabla\mathbf{r})^t(W_{\mathbf{F}})\mathbf{k}. \tag{10.32}$$

We see, from eqn (10.30), part 1, that G is stationary at any solution to our problem, i.e., $G_{\mathbf{e}} = \mathbf{0}$.

The second derivative is

$$\ddot{\sigma} = G_{\mathbf{e}} \cdot \ddot{\mathbf{e}} + (G_{\mathbf{e}})^{\cdot} \cdot \dot{\mathbf{e}}, \tag{10.33}$$

where

$$(G_{\mathbf{e}})^{\cdot} = (\nabla \mathbf{r})^t (W_{\mathrm{FF}}[(\nabla \mathbf{r})\dot{\mathbf{e}} \otimes \mathbf{k}])\mathbf{k}. \tag{10.34}$$

Problem

> Reduce this to $\ddot{\sigma} = G_{\mathbf{e}} \cdot \ddot{\mathbf{e}} + (\nabla \mathbf{r})\dot{\mathbf{e}} \cdot \{\mathbf{A}(\mathbf{F}, \mathbf{k})\}(\nabla \mathbf{r})\dot{\mathbf{e}}$, where $\mathbf{A}(\mathbf{F}, \mathbf{k})$ is the acoustic tensor based on the unit vector $\mathbf{k}$.

The linear space of two-vectors is a convex set. As such, it contains the straight-line path $\mathbf{e}(u) = (1 - u)\mathbf{e}_1 + u\mathbf{e}_2$, with $u \in [0, 1]$, for any pair $\mathbf{e}_1$ and $\mathbf{e}_2$ of distinct two-vectors. That is, every vector can be expressed a convex combination of two given vectors. On this path we have $\ddot{\mathbf{e}} = \mathbf{0}$ and

$$\ddot{\sigma} = (\nabla \mathbf{r})\dot{\mathbf{e}} \cdot \{\mathbf{A}(\mathbf{F}, \mathbf{k})\}(\nabla \mathbf{r})\dot{\mathbf{e}}, \tag{10.35}$$

with $\dot{\mathbf{e}} = \mathbf{e}_2 - \mathbf{e}_1 (\neq \mathbf{0})$.

Recall that strong ellipticity or material stability is the requirement, for incompressible materials, that $\mathbf{a} \cdot \mathbf{A}(\mathbf{F}, \mathbf{b})\mathbf{a} > 0$ for all non-zero $\mathbf{a}$ and any unit vector $\mathbf{b}$, such that $\mathbf{a} \cdot \mathbf{F}^*\mathbf{b} = 0$. Picking $\mathbf{a} = (\nabla \mathbf{r})\dot{\mathbf{e}}$ and $\mathbf{b} = \mathbf{k}$, and invoking eqn (10.17), we find that $\mathbf{a} \cdot \mathbf{F}^*\mathbf{b} = (\nabla \mathbf{r})\dot{\mathbf{e}} \cdot (\alpha \mathbf{n})$, which vanishes identically. Accordingly, strong ellipticity implies that $\ddot{\sigma} > 0$ for all $u \in [0, 1]$. Integration then yields

$$\dot{\sigma}(u) = \dot{\sigma}(0) + \int_0^u \ddot{\sigma}(x)dx > \dot{\sigma}(0) \tag{10.36}$$

if $u > 0$ and, hence,

$$\sigma(1) = \sigma(0) + \int_0^1 \dot{\sigma}(u)du > \sigma(0) + \dot{\sigma}(0), \tag{10.37}$$

i.e.,

$$G(\mathbf{e}_2) - G(\mathbf{e}_1) > (\mathbf{e}_2 - \mathbf{e}_1) \cdot G_{\mathbf{e}}(\mathbf{e}_1). \tag{10.38}$$

This means that $G(\mathbf{e})$ is a convex function. Of course, we can interchange $\mathbf{e}_1$ and $\mathbf{e}_2$, and repeat the foregoing argument, obtaining

$$G(\mathbf{e}_1) - G(\mathbf{e}_2) > (\mathbf{e}_1 - \mathbf{e}_2) \cdot G_{\mathbf{e}}(\mathbf{e}_2). \tag{10.39}$$

Adding these inequalities, we conclude that

$$[G_e(\mathbf{e}_2) - G_e(\mathbf{e}_1)] \cdot (\mathbf{e}_2 - \mathbf{e}_1) > 0, \quad \text{for all} \quad \mathbf{e}_2 \neq \mathbf{e}_1. \tag{10.40}$$

Suppose now that there are two solutions, $\mathbf{e}_1$ and $\mathbf{e}_2$, say, to the stationarity problem. Then $G_e(\mathbf{e}_2) - G_e(\mathbf{e}_1)$ is manifestly zero, and the only possibility consistent with (10.40) is $\mathbf{e}_2 = \mathbf{e}_1$; the solution $\mathbf{e}$ is unique. Equation (10.30), part 2, then furnishes a unique constraint pressure.

Beyond this, if $\mathbf{e}_1$, say, is the solution, then eqn (10.38) yields $G(\mathbf{e}_2) > G(\mathbf{e}_1)$ for any $\mathbf{e}_2$ not equal to $\mathbf{e}_1$. We conclude that the solution to eqn (10.30), part 1, minimizes the energy relative to any alternative value of $\mathbf{e}$.

10.4 Isotropic materials

By far the majority of applications of membrane theory concern isotropic, incompressible materials, and so we confine our further attention to this important case. With reference to eqn (4.41) and Chapter 6, we may write $\mathbf{P} = \mathbf{FS}$, with

$$\mathbf{S} = \sum_{i=1}^{3} \lambda_i^{-1}(\partial\omega/\partial\lambda_i - q/\lambda_i)\mathbf{u}_i \otimes \mathbf{u}_i, \tag{10.41}$$

where $\omega(\lambda_1, \lambda_2, \lambda_3)$ is the (extended) strain–energy function, written in terms of the principal stretches, and $\mathbf{u}_i$ are the associated (orthonormal) principal axes. As $\mathbf{F}$ is invertible, the plane stress condition eqn (10.29), part 2, is equivalent to the statement

$$\mathbf{Sk} = \mathbf{0}. \tag{10.42}$$

This implies that $\mathbf{k}$ is an eigenvector of $\mathbf{S}$, with eigenvalue zero. We may, therefore, identify $\mathbf{k}$ with $\mathbf{u}_3$, say, and conclude that

$$q = \lambda_3\partial\omega/\partial\lambda_3 = (\lambda_1\lambda_2)^{-1}\partial\omega/\partial\lambda_3, \tag{10.43}$$

which is seen to be equivalent to eqn (10.30), part 2, and where isochoricity has been imposed in the form $\lambda_1\lambda_2\lambda_3 = 1$. Because the $\mathbf{u}_i$ are orthogonal, the $\mathbf{u}_\alpha$ lie in the plane Ω; accordingly, from eqn (10.15),

$$\lambda_1\mathbf{v}_1 = (\nabla\mathbf{r})\mathbf{u}_1 \quad \text{and} \quad \lambda_2\mathbf{v}_2 = (\nabla\mathbf{r})\mathbf{u}_2, \tag{10.44}$$

implying that the principal vectors $\mathbf{v}_\alpha$ are tangential to ω. This is enough to conclude that $\mathbf{v}_3$ is perpendicular to the deformed surface and, thus, aligned with its unit normal $\mathbf{n}$. Here, of course, $\mathbf{v}_i$ are the eigenvectors of the left stretch tensor $\mathbf{V}$. We suppose, without loss of generality, that $\{\mathbf{u}_i\}$ and $\{\mathbf{v}_i\}$ are right-handed triads. The positivity of the scalar triple product $J = [\mathbf{Fu}_1, \mathbf{Fu}_2, \mathbf{Fu}_3]$ then furnishes $\mathbf{v}_3 = \mathbf{n}$. Finally, eqn (10.15) delivers

$$\mathbf{d} = \lambda_3 \mathbf{n}, \tag{10.45}$$

implying that $\alpha = \lambda_3^{-1}$ and $\mathbf{e} = \mathbf{0}$ in eqn (10.22); this is the unique director field.

Let

$$\hat{\omega}(\lambda_1, \lambda_2) = \omega(\lambda_1, \lambda_2, (\lambda_1\lambda_2)^{-1}). \tag{10.46}$$

Then,

$$\begin{aligned}\partial\hat{\omega}/\partial\lambda_1 &= \partial\omega/\partial\lambda_1 + (\partial\omega/\partial\lambda_3)\partial\lambda_3/\partial\lambda_1 \\ &= \partial\omega/\partial\lambda_1 - \lambda_1^{-1}\lambda_3\partial\omega/\partial\lambda_3 \\ &= \partial\omega/\partial\lambda_1 - \lambda_1^{-1}q.\end{aligned} \tag{10.47}$$

Likewise,

$$\partial\hat{\omega}/\partial\lambda_2 = \partial\omega/\partial\lambda_2 - \lambda_2^{-1}q. \tag{10.48}$$

It follows that

$$\mathbf{S} = \sum_{\alpha=1}^{2} \lambda_\alpha^{-1}\partial\hat{\omega}/\partial\lambda_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha \tag{10.49}$$

and

$$\mathbf{P} = \sum_{\alpha=1}^{2} \partial\hat{\omega}/\partial\lambda_\alpha \mathbf{v}_\alpha \otimes \mathbf{u}_\alpha. \tag{10.50}$$

Note that $\mathbf{P} = \mathbf{P1}$ because we have already solved $\mathbf{Pk} = \mathbf{0}$. Also, we have used the traction conditions at the major surfaces to evaluate the constraint pressure a priori, and so the membrane problem does not involve a Lagrange multiplier.

Often the h-multiplied version of the membrane problem is preferred. This is

$$Div_{\parallel}(h\mathbf{P}) + \alpha p\mathbf{n} = \mathbf{0}, \tag{10.51}$$

where p is the actual pressure, apart from an error of order $o(h)$, $\alpha = \lambda_1\lambda_2$ and

$$h\mathbf{P} = \sum_{\alpha=1}^{2} \partial U/\partial\lambda_\alpha \mathbf{v}_\alpha \otimes \mathbf{u}_\alpha, \tag{10.52}$$

where

$$U(\lambda_1, \lambda_2) = h\hat{\omega}(\lambda_1, \lambda_2) \tag{10.53}$$

is the strain energy per unit area of Ω.

Before proceeding to an example, we pause to re-write the equations in yet another, arguably more convenient, form. To this end, note that

$$hP = hP1 = h(PE_\alpha) \otimes E_\alpha = p_\alpha \otimes E_\alpha, \tag{10.54}$$

where

$$p_\alpha = hPE_\alpha = hP_{i\alpha} e_i. \tag{10.55}$$

Accordingly,

$$Div_{\|}(hP) = hP_{i\alpha,\alpha} e_i = p_{\alpha,\alpha}, \tag{10.56}$$

and the equation to be solved is

$$p_{\alpha,\alpha} + p\alpha n = 0. \tag{10.57}$$

Physically, the stress vectors p_α are the force resultants (forces per unit length) transmitted across the material lines on which the u_α are constant.

10.5 Axially symmetric deformations of a cylindrical membrane

Consider a reference configuration of a membrane in the shape of a right circular cylinder. We can regard this configuration as a mapping from an initial plane, as in the previous subsection, by writing $\mathbf{u} = u_\alpha \mathbf{E}_\alpha$ with $u_1 = z$ and $u_2 = R\theta$, where z is the axial coordinate along the axis of the tube, $\theta \in [0, 2\pi)$ is the azimuthal angle, assuming constant values on the generators of the tube, and R is the (constant) tube radius. As before, we identify $\mathbf{E}_3$ with $\mathbf{k}$, the unit normal to the plane (Figures 10.3a–d).

We suppose the deformed membrane to be a surface of revolution, parametrized in the form

$$\mathbf{r}(\mathbf{u}) = r(z)\mathbf{e}_r(\theta) + \xi(z)\mathbf{e}_1, \tag{10.58}$$

where $\mathbf{e}_r(\theta) = \cos\theta \mathbf{e}_2 + \sin\theta \mathbf{e}_3$ (note carefully that the subscript labels do not conform to the common convention). The axis of symmetry is directed along $\mathbf{e}_1$, and we impose $\{\mathbf{e}_i\} = \{\mathbf{E}_A\}$. Here, $r(z)$ and $\xi(z)$ are the radius and axial coordinate, after deformation, of the material circle defined by $z = const.$ on the reference cylinder. The latter is recovered by putting $r(z) = R$ and $\xi(z) = z$ (Figure 10.3).

To construct the membrane deformation gradient, we use

$$(\nabla\mathbf{r})d\mathbf{u} = d\mathbf{r} = [r'(z)\mathbf{e}_r(\theta) + \xi'(z)\mathbf{e}_1]dz + r(z)\mathbf{e}_\theta(\theta)d\theta, \tag{10.59}$$

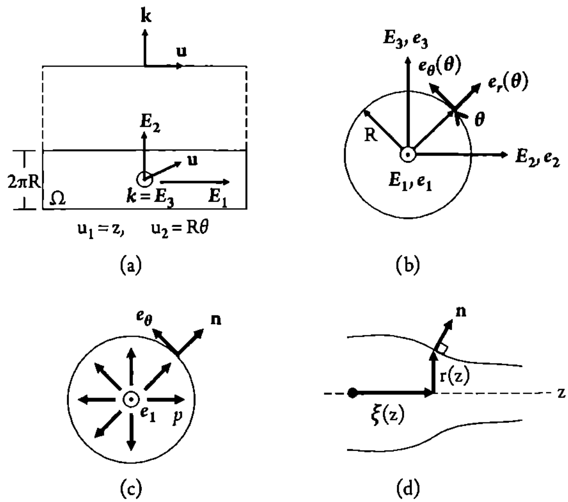

Figure 10.3 Geometry of reference and deformed surface. (a) Plane isometric to reference cylinder. (b) Geometry of reference cylinder. (c) Section of deformed surface of revolution. (d) Meridian of deformed surface.

where $\mathbf{e}_\theta(\theta) = \mathbf{e}_r'(\theta) = \mathbf{e}_1 \times \mathbf{e}_r(\theta)$. Noting that $dz = du_1 = \mathbf{E}_1 \cdot d\mathbf{u}$ and $d\theta = R^{-1}du_2 = R^{-1}\mathbf{E}_2 \cdot d\mathbf{u}$, we conclude, in accordance with eqn (10.16), that

$$\nabla \mathbf{r} = (r'\mathbf{e}_r + \xi'\mathbf{e}_1) \otimes \mathbf{E}_1 + (r/R)\mathbf{e}_\theta \otimes \mathbf{E}_2, \tag{10.60}$$

which immediately delivers

$$\nabla \mathbf{r} = \sum_{\alpha=1}^{2} \lambda_\alpha \mathbf{v}_\alpha \otimes \mathbf{u}_\alpha, \tag{10.61}$$

with $\mathbf{u}_1 = \mathbf{E}_1$, $\mathbf{u}_2 = \mathbf{E}_2$, and

$$\lambda_1 \mathbf{v}_1 = r'\mathbf{e}_r + \xi'\mathbf{e}_1, \quad \lambda_2 \mathbf{v}_2 = (r/R)\mathbf{e}_\theta. \tag{10.62}$$

Hence, the principal stretches:

$$\lambda_1 = \sqrt{(r')^2 + (\xi')^2} \quad \text{and} \quad \lambda_2 = r/R. \tag{10.63}$$

We also have the stress vectors:

$$\mathbf{p}_\alpha = \left(\sum_{\beta=1}^{2} \partial U/\partial \lambda_\beta \mathbf{v}_\beta \otimes \mathbf{u}_\beta \right) \mathbf{E}_\alpha, \tag{10.64}$$

so that

$$\mathbf{p}_1 = \partial U/\partial\lambda_1\mathbf{v}_1 \quad \text{and} \quad \mathbf{p}_2 = \partial U/\partial\lambda_2\mathbf{v}_2, \tag{10.65}$$

yielding

$$\mathbf{p}_{\alpha,\alpha} = \frac{\partial}{\partial z}(\lambda_2 t_1\mathbf{v}_1) + \frac{\partial}{R\partial\theta}(\lambda_1 t_2\mathbf{v}_2), \tag{10.66}$$

where

$$t_1 = \lambda_2^{-1}\partial U/\partial\lambda_1 \quad \text{and} \quad t_2 = \lambda_1^{-1}\partial U/\partial\lambda_2. \tag{10.67}$$

Problem

Show that the t_α are the principal Cauchy stress resultants, i.e., the eigenvalues of $h^*\mathbf{T}$, where $\mathbf{T}$ is the value of the Cauchy stress at the midsurface and h^* is the thickness of the membrane in its deformed configuration.

Using eqns (10.62) and (10.65) we derive

$$\frac{\partial}{\partial z}(\lambda_2 t_1\mathbf{v}_1) = (\lambda_2\lambda_1^{-1}t_1)'\lambda_1\mathbf{v}_1 + \lambda_2\lambda_1^{-1}t_1(r''\mathbf{e}_r + \xi''\mathbf{e}_1) \quad \text{and}$$

$$\frac{\partial}{\partial\theta}(\lambda_1 t_2\mathbf{v}_2) = \lambda_1 t_2\mathbf{e}_\theta'(\theta) = -\lambda_1 t_2\mathbf{e}_r, \tag{10.68}$$

and eqn (10.57) reduces to

$$-R^{-1}\lambda_1 t_2\mathbf{e}_r + (\lambda_2\lambda_1^{-1}t_1)'\lambda_1\mathbf{v}_1 + \lambda_2\lambda_1^{-1}t_1(r''\mathbf{e}_r + \xi''\mathbf{e}_1) + p\lambda_1\lambda_2\mathbf{n} = \mathbf{0}. \tag{10.69}$$

Due to of our unorthodox labelling of axes, the *exterior* unit normal to the deformed membrane is obtained using $\lambda_1\lambda_2\mathbf{n} = -\lambda_1\mathbf{v}_1 \times \lambda_2\mathbf{v}_2$. Projecting eqn (10.69) onto $\mathbf{e}_1$, we then find that

$$(\lambda_2\lambda_1^{-1}t_1\xi')' - pr'(r/R) = 0, \tag{10.70}$$

which may be integrated, in the case of a uniform inflation pressure, to yield

$$\lambda_1^{-1}\partial U/\partial\lambda_1\xi' = \frac{1}{2}pr^2/R + C, \tag{10.71}$$

where C is a constant.

Problem

Prove that this constant is proportional to the axial force acting on a cross section of the cylinder. The result is, therefore, an elementary consequence of axial force equilibrium.

Projection of eqn (10.69) onto the tangent, $\mathbf{v}_1$, to the meridian of the deformed membrane yields

$$-(r'/R)t_2 + (\lambda_2\lambda_1^{-1}t_1)'\lambda_1 + \lambda_2\lambda_1^{-1}t_1\lambda_1^{-1}(r'r'' + \xi'\xi'') = 0. \tag{10.72}$$

The second term is the same as $(\lambda_2 t_1)' - \lambda_2\lambda_1^{-1}t_1\lambda_1'$ and the final term in parentheses is $\lambda_1\lambda_1'$. The equation thus reduces to $(\lambda_2 t_1)' = \lambda_2' t_2$, or

$$(rt_1)' = t_2 r'. \tag{10.73}$$

Together with eqn (10.71), this provides a system for the determination of $r(z)$ and $\xi(z)$.

Problem

Consider the problem of an unpressurized membrane of length $2L$ mounted on parallel rings of radius R at $z = \pm L$. An axial force $\pm F$ is applied to these rings. The membrane is composed of a neo-Hookean material (In Problem no, 2 of Section 9.1 you proved that this material satisfies strong ellipticity). Compute the relation between this force and the axial half-length, l, of the membrane, i.e., $l = \xi(L)$.

To proceed, use symmetry to justify the assumption that $\xi(z)$ is an odd function; i.e., $\xi(-z) = -\xi(z)$. Then, $\xi(0) = 0$. Next, observe that eqn (10.63), part 1, implies the existence of an angle $\phi(z)$ such that

$$r' = \lambda_1 \sin\phi, \quad \xi' = \lambda_1 \cos\phi. \tag{10.74}$$

Thus, $\tan\phi = dr/d\xi$, implying that ϕ is the angle made by the tangent to the meridian with the symmetry axis. From Figure 10.4, we infer that the tangent to the meridian vanishes at the throat of the membrane and, hence, that $\phi(0) = 0$.

From eqn (10.71) we have $\lambda_1^{-1}\partial U/\partial\lambda_1\xi' = F/2\pi R$, or

$$F/2\pi R = \partial U/\partial\lambda_1 \cos\phi, \tag{10.75}$$

whereas eqn (10.73) furnishes $(\partial U/\partial\lambda_1)' = \lambda_2'\lambda_1^{-1}\partial U/\partial\lambda_2$. Expanding the derivative on the left and solving for λ_1', we derive

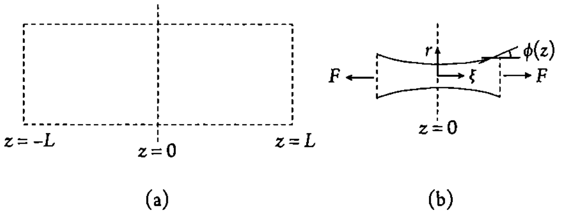

Figure 10.4 Reference and deformed configurations of the membrane

$$\lambda_1' = (\partial^2 U/\partial\lambda_1^2)^{-1} R^{-1} \sin\phi(\partial U/\partial\lambda_2 - \lambda_1 \partial^2 U/\partial\lambda_1\partial\lambda_2), \tag{10.76}$$

where use has been made of eqn (10.74), part 1, in the form

$$\lambda_2' = R^{-1}\lambda_1 \sin\phi. \tag{10.77}$$

Equations eqns (10.75)–(10.77) provide a differential-algebraic system to be solved for the functions $\lambda_1(z)$, $\lambda_2(z)$ and $\phi(z)$, subject to the boundary conditions $\phi(0) = 0$ and $\lambda_2(L) = 1$.

To solve this system use a shooting scheme, with Euler *backward* differencing commencing at $z/L = 1$. This entails guessing the missing boundary value $\phi_L = \phi(L)$ and adjusting it until a solution having $\phi(0) = 0$ is achieved. Nondimensionalize the problem using the length L and the shear modulus of the material. Finally, compute

$$l = \int_0^L \lambda_1 \cos\phi dz \tag{10.78}$$

to obtain the value of l corresponding to the assigned value of the force. Can you plot the shape of the deformed membrane, i.e., r vs. ξ?

Re-analyze this problem for the bio-elastic material defined by eqn (7.38). Does this material satisfy strong ellipticity? Consider various (positive) values of γ.

A useful result follows from eqn (10.73), which we write in the form

$$\lambda_2'\partial U/\partial\lambda_2 = \lambda_1(\partial U/\partial\lambda_1)' = (\lambda_1\partial U/\partial\lambda_1)' - \lambda_1'\partial U/\partial\lambda_1. \tag{10.79}$$

Noting that $\lambda_1'\partial U/\partial\lambda_1 + \lambda_2'\partial U/\partial\lambda_2 = U'(\lambda_1, \lambda_2)$ for uniform materials, we find (see Pipkin, 1968) that this integrates to $H = const.$, where

$$H(\lambda_1, \lambda_2) = U(\lambda_1, \lambda_2) - \lambda_1\partial U/\partial\lambda_1. \tag{10.80}$$

This furnishes a simple check on the accuracy of solutions.

Problem

Using the results of the previous exercise, plot H as a function of z and confirm that it is indeed constant.

10.6 Bulging of a cylinder

A party balloon may be idealized as a long cylindrical tube. This is sealed at its ends and pressurized, but we assume the tube to be long enough that the details of the deformation near these ends can be safely ignored. One typically observes that the cylinder deforms, roughly,

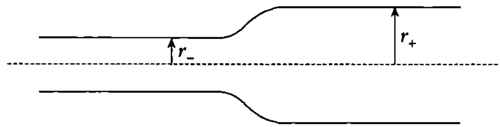

Figure 10.5 Bulging of a cylinder

into another cylinder, with radius depending on the pressure, until a certain threshold pressure is reached, at which point a bulge starts to form at one end. This bulge propagates down the length of the cylinder at a more-or-less fixed value of the pressure p^*, say, until it consumes the entire balloon. Thereafter, an increase in pressure again produces a roughly cylindrical membrane, with radius again depending on the pressure. During the bulge-propagation phase at pressure p^*, the membrane is deformed into two coexistent cylinders, separated by a transition region in which the radius varies with the axial coordinate (Figure 10.5). We use membrane theory to derive a simple model of this interesting phenomenon.

With reference to the figure, we seek a solution in which the membrane has deformed radii $r^{\pm}$ on either side of a transition region. We refer to these uniform states as *phases*. In either of the phases we have $r' = 0$ and, therefore, $\lambda_1 = \xi'$, assuming $\xi(z)$ to be an increasing function. An elementary balance of axial forces yields $2\pi R\partial U/\partial\lambda_1 = p\pi r^2$, or

$$\partial U/\partial\lambda_1 = \frac{1}{2}pr^2/R, \tag{10.81}$$

and comparison with eqn (10.71) indicates that $C = 0$. We recast the latter equation, which holds throughout the membrane (including the transition region) in the form

$$\partial U/\partial\lambda_1\xi' = \frac{1}{2}pR\lambda_1\lambda_2^2. \tag{10.82}$$

In any deformation that maps the cylinder to a cylinder ($r' = 0$) this reduces to

$$\partial U/\partial\lambda_1 = \frac{1}{2}R(p\lambda_2^2), \tag{10.83}$$

which is just eqn (10.81). In this case elementary statics also provides $t_2 = pr$, or

$$\partial U/\partial\lambda_2 = \frac{1}{2}R(2p\lambda_1\lambda_2). \tag{10.84}$$

The total strain energy stored in a purely cylindrical deformation of a membrane of initial length L is $2\pi RLU$. As the initial volume is πR^2L, the strain energy per unit of initial volume is

$$E = (2/R)U. \tag{10.85}$$

The volume contained by the deformed cylinder, per unit of initial volume, is $v = \pi r^2\lambda_1 L/\pi R^2 L$, or

$$\nu = \lambda_1 \lambda_2^2. \tag{10.86}$$

Consider a uniform equilibrium deformation carrying the membrane from a cylinder of radius r^- to a cylinder of radius r^+. The induced change of strain energy is

$$E^+ - E^- = 2/R \int_-^+ dU = 2/R \int_-^+ \partial U/\partial \lambda_1 d\lambda_1 + \partial U/\partial \lambda_2 d\lambda_2, \tag{10.87}$$

where the integration limits refer to the two states. Using eqns (10.83) and (10.84), we reduce this to

$$E^+ - E^- = \int_-^+ 2p\lambda_1\lambda_2 d\lambda_2 + p\lambda_2^2 d\lambda_1. \tag{10.88}$$

Thus, from eqn (10.86),

$$E^+ - E^- = \int_{\nu^-}^{\nu^+} p(\nu) d\nu. \tag{10.89}$$

To determine $p(\nu)$ for a given membrane, we select the function $U(\lambda_1, \lambda_2)$ and solve eqns (10.83) and (10.84) for λ_1, say. We then solve the same system for p in terms of λ_2, and plot p against $\nu = \lambda_1\lambda_2^2$. Alternatively, fix p and solve eqns (10.83) and (10.84) for λ_1 and λ_2, and then plot p against ν.

Problem

Carry out the details in the case of Ogden's (1997) strain–energy function

$$U(\lambda_1, \lambda_2) = \sum_{i=1}^{3} \mu_i I(\lambda_1, \lambda_2; \alpha_i), \tag{10.90}$$

where

$$I(\lambda_1, \lambda_2; \alpha) = \alpha^{-1}[\lambda_1^\alpha + \lambda_2^\alpha + (\lambda_1\lambda_2)^{-\alpha} - 3], \tag{10.91}$$

with

$$\alpha_1 = 1.3, \quad \alpha_2 = 5.0, \quad \alpha_3 = -2.0 \tag{10.92}$$

and

$$\mu_2/\mu_1 = 2.01 \times 10^{-3}, \quad \mu_3/\mu_1 = -1.59 \times 10^{-2}. \tag{10.93}$$

Plot $Rp(\nu)/\mu_1$ against ν.

Note: Ball (1977) has shown that this strain–energy function satisfies a condition known as polyconvexity, which we shall discuss later. Furthermore, polyconvexity is sufficient for strong ellipticity, and so Ogden's (1997) function satisfies the conditions we have assumed in the course of deriving membrane theory.

Returning to the coexistent phase problem, recall that eqn (10.82) holds throughout the membrane. If p^* is the pressure associated with the two-phase solution (Figure 10.6), then in the uniform phases we have

$$(\lambda_1 \partial U/\partial\lambda_1)^{\pm} = (R/2)p^* v^{\pm}, \tag{10.94}$$

whereas eqn (10.80) furnishes

$$(2/R)(\lambda_1 \partial U/\partial\lambda_1 - U)^{+} = (2/R)(\lambda_1 \partial U/\partial\lambda_1 - U)^{-}. \tag{10.95}$$

These combine to give

$$E^{+} - E^{-} = p^*(v^{+} - v^{-}). \tag{10.96}$$

Comparison with eqn (10.89) then furnishes the means to calculate p^*. Thus,

$$p^*(v^{+} - v^{-}) = \int_{v^-}^{v^+} p(v)dv. \tag{10.97}$$

This is the famous Maxwell equal-area rule for phase coexistence: The left-hand side is the area of a rectangle of base $v^+ - v^-$ and height p^* and, of course, the right-hand side is the area under the graph of $p(v)$ between v^- and v^+. One simply adjusts the value of p^* accordingly. In practice, one finds a unique pair (v^+, v^-) for which this construction is possible, and simply reads off the associated value of p^* from the graph (Figure 10.6).

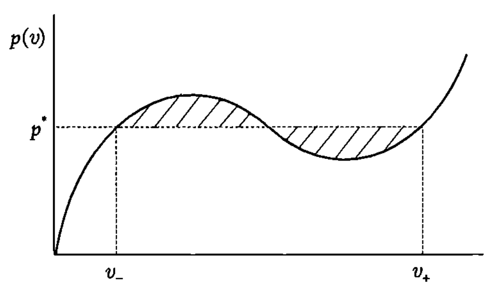

Figure 10.6 Pressure–volume characteristic for purely cylindrical deformations. Shaded lobes have equal areas

Problem

Complete the detailed analysis using results obtained in the previous exercise.

Further discussion of this problem, and other interesting examples of phase coexistence in membranes, may be found in the references cited. We will take up Maxwell's rule again, in another context, in the next chapter.

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11

Stability and the energy criterion

In this chapter we elaborate on the notion of stability of equilibria with particular emphasis on the energy criterion for conservative problems, i.e., problems in which the loading may be associated with a potential energy, as in Problem no. 2 of Section 3.2. We show that for an equilibrium state of a conservative system to be stable, it is necessary that it furnish a minimum of the potential energy consisting of the strain energy and load potential. This subject has a foundation in thermodynamics (see the paper by Ericksen (1966)). However, in keeping with the theme of this book, we rely on a purely mechanical argument.

11.1 The energy norm

An immediate issue we must face is the matter of how to quantify stability and instability. Intuitively we imagine that a state of equilibrium, say, is stable or unstable according as the size of any disturbance to this state remains bounded, or otherwise, as time evolves. This notion of size indicates that we must incorporate an appropriate norm into the definition of stability/instability. In this respect, the study of stability for continuous systems is far more intricate than it is for discrete or finite-dimensional systems. For, in the latter case all norms are equivalent, whereas in the former there is no such equivalence. One can construct examples where a given state can be stable as measured by a given norm, but unstable in terms of another. See the book by Como and Grimaldi (1995) and the treatise by Knops and Wilkes (1973). Thus, it is immediately clear that no state can be stable without qualification, i.e., the judgment as to stability or otherwise is inherently norm dependent. Therefore, it becomes necessary to choose a norm judiciously, to ensure the greatest degree of contact with the phenomena at hand.

A choice of norm that presents itself almost immediately is the so-called *energy norm* advocated by Mikhlin (1965) and subsequently studied by Como and Grimaldi (1995). This meets the formal definition of a norm while conferring a meaning that is intrinsic to the problems considered. We develop its description in stages, starting with some elementary observations. First, recall the definition eqn (9.6) of the tensor of elastic moduli; namely,

$$\mathcal{M} = W_{\mathbf{FF}}, \tag{11.1}$$

Finite Elasticity Theory. David J. Steigmann.

from which it follows that $\mathcal{M} = \mathbf{P}_{\mathbf{F}}$. In other words, on any path $\mathbf{F}(\mu)$, we have $\dot{\mathbf{P}} = \mathcal{M}[\dot{\mathbf{F}}]$. In the same way we can define the strain-dependent moduli

$$\mathcal{C} = \tilde{W}_{\mathbf{EE}}, \tag{11.2}$$

where $\mathbf{E}$ is the Lagrange strain, i.e.,

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^t\mathbf{F} - \mathbf{I}), \tag{11.3}$$

and $\tilde{W}(\mathbf{E}) = \hat{W}(2\mathbf{E} + \mathbf{I})$ is the strain-energy function expressed as a function of strain. Accordingly, $\mathcal{C} = \mathbf{S}_{\mathbf{E}}$, the derivative of the 2nd Piola-Kirchhoff stress $\mathbf{S} = \tilde{W}_{\mathbf{E}}$ with respect to $\mathbf{E}$. Here we adopt the convention that the derivative $\tilde{W}_{\mathbf{E}}$ is an element of *Sym*. Like $\mathcal{M}$, the tensor of strain-dependent moduli possesses the major symmetry $\mathcal{C}^t = \mathcal{C}$. Unlike $\mathcal{M}$, it also possesses the minor symmetries $\mathbf{A}^t \cdot \mathcal{C}[\mathbf{B}] = \mathbf{A} \cdot \mathcal{C}[\mathbf{B}]$ and $\mathbf{A} \cdot \mathcal{C}[\mathbf{B}^t] = \mathbf{A} \cdot \mathcal{C}[\mathbf{B}]$. Further, $\dot{\mathbf{S}} = \mathcal{C}[\dot{\mathbf{E}}]$. Applying the chain rule to the relation $\mathbf{P} = \mathbf{FS}$ thus furnishes

$$\mathcal{M}[\dot{\mathbf{F}}] = \dot{\mathbf{F}}\mathbf{S} + \frac{1}{2}\mathbf{F}\mathcal{C}[\dot{\mathbf{F}}^t\mathbf{F} + \mathbf{F}^t\dot{\mathbf{F}}]. \tag{11.4}$$

Due to the minor symmetry of $\mathcal{C}$ and because $\dot{\mathbf{F}}$ is arbitrary, we then have

$$\mathcal{M}[\mathbf{A}] = \mathbf{AS} + \mathbf{F}\mathcal{C}[\mathbf{F}^t\mathbf{A}] \tag{11.5}$$

for all $\mathbf{A}$.

Consider a deformation path with $\mathbf{F}(0) = \mathbf{I}$. Then for small μ,

$$W(\mathbf{F}(\mu)) = W(\mathbf{I}) + \mu\mathbf{P}_0 \cdot \dot{\mathbf{F}}_0 + \frac{1}{2}\mu^2\mathcal{M}_0[\dot{\mathbf{F}}_0] \cdot \dot{\mathbf{F}}_0 + o(\mu^2), \tag{11.6}$$

where $\mathbf{P}_0$, $\mathcal{M}_0$, and $\dot{\mathbf{F}}_0$, respectively, are the values of $\mathbf{P}$, $\mathcal{M}$, and $\dot{\mathbf{F}}$ at $\mathbf{F} = \mathbf{I}$. Without loss of generality we can impose $W(\mathbf{I}) = 0$, as the value of $W(\mathbf{I})$ may be adjusted arbitrarily at any material point without affecting the values of measurable quantities. Suppose that $\mathbf{P}_0 (= \mathbf{S}_0)$ vanishes. Then,

$$W(\mathbf{F}(\mu)) = \frac{1}{2}\mu^2\mathcal{C}_0[\dot{\mathbf{F}}_0] \cdot \dot{\mathbf{F}}_0 + o(\mu^2), \tag{11.7}$$

where $\mathcal{C}_0$ is the value of $\mathcal{C}$ at $\mathbf{E} = \mathbf{0}$.

Let $\chi(\mu)$ be the deformation associated with gradient $\mathbf{F}(\mu)$, and let $\mathbf{v}(\mathbf{x})$ be the value of $\dot{\chi}$ at $\mu = 0$. Then, $\dot{\mathbf{F}}_0 = \nabla\mathbf{v}$ and

$$W(\mathbf{F}(\mu)) = \frac{1}{2}\mu^2\mathcal{C}_0[\varepsilon] \cdot \varepsilon + o(\mu^2), \tag{11.8}$$

where

$$\varepsilon = Sym(\nabla\mathbf{v}) \tag{11.9}$$

is the *infinitesimal strain* tensor. This is the strain measure used in linear elasticity theory. We have invoked the minor symmetries of $\mathcal{C}_0$ to replace $\nabla \mathbf{v}$ by its symmetric part. It is customary in that theory to take $\mathcal{C}_0$ to be positive definite; that is,

$$\mathcal{C}_0[\mathbf{A}] \cdot \mathbf{A} > 0 \tag{11.10}$$

for all $\mathbf{A}$ with non-zero symmetric part ($Sym\mathbf{A} \neq \mathbf{0}$). We observe in passing that $\mathcal{M}_0[\mathbf{A}] \cdot \mathbf{A} = \mathcal{C}_0[\mathbf{A}] \cdot \mathbf{A}$ under our hypotheses, and so our assumption implies material stability of the reference configuration, i.e., strong ellipticity at $\mathbf{F} = \mathbf{I}$.

Problem

Prove this claim; that is, show that if $\mathbf{A} = \mathbf{a} \otimes \mathbf{b} \neq \mathbf{0}$, then $Sym\mathbf{A} \neq \mathbf{0}$.

Crucially, eqn (11.10) implies that $\mathcal{C}_0$ is bounded below in the sense that

$$\mathcal{C}_0[\varepsilon] \cdot \varepsilon \geq \lambda \left|\varepsilon\right|^2, \tag{11.11}$$

where $\lambda(\mathbf{x})$ is a positively valued scalar field. To see this we use the spectral theorem for symmetric matrices. Let ξ be a 6-vector consisting of the components of ε on any orthonormal basis. Let S be the symmetric 6×6 matrix consisting of the components of $\mathcal{C}_0$ on the same basis. Then, from the spectral theorem,

$$\xi' S \xi = \sum_{i=1}^{6} \lambda_i (\xi' s_i)^2 \geq \lambda \sum_{i=1}^{6} (\xi' s_i)^2 = \lambda \left|\xi\right|^2, \tag{11.12}$$

where s_i are the (orthonormal) eigenvectors of S, λ_i are the associated eigenvalues and $\lambda = \min\{\lambda_i\}$. The latter is strictly positive because S is positive definite, and of course this is just (11.11).

We define $\|\mathbf{v}\|$ by

$$\|\mathbf{v}\|^2 = \int_\kappa \mathcal{C}_0[\varepsilon] \cdot \varepsilon dv \quad \text{with} \quad \varepsilon = Sym(\nabla \mathbf{v}). \tag{11.13}$$

This is called the *energy norm*. It is clearly intrinsic to the material at hand. To verify that it is in fact a norm, we first use eqn (11.11) to infer that

$$\|\mathbf{v}\|^2 \geq \bar{\lambda} \int_\kappa \left|\varepsilon\right|^2 dv, \tag{11.14}$$

where $\bar{\lambda} = \min_{x \in \kappa} \lambda(\mathbf{x})$. This exists, and is strictly positive, because $\lambda(\mathbf{x})$ is positive and continuous—assuming that $\mathcal{C}_0(\mathbf{x})$ is likewise continuous, and because κ is compact (i.e., closed and bounded) in three-space (see Palis and deMelo, 1982).

Korn's inequality asserts the existence of a positive constant, k say, such that

$$k \int_{\kappa} |\varepsilon|^2 \, dv \geq \int_{\kappa} |\nabla \mathbf{v}|^2 \, dv, \tag{11.15}$$

provided that $\mathbf{v}(\mathbf{x})$ vanishes on some non-empty subset of the boundary $\partial\kappa$. Poincaré's inequality is the assertion that

$$\int_{\kappa} |\nabla \mathbf{v}|^2 \, dv \geq c \int_{\kappa} |\mathbf{v}|^2 \, dv \tag{11.16}$$

for some positive constant c, under the same restriction on $\mathbf{v}(\mathbf{x})$. Accordingly, there is a positive constant a such that

$$\|\mathbf{v}\|^2 \geq a \int_{\kappa} |\mathbf{v}|^2 \, dv. \tag{11.17}$$

Proofs of the Korn and Poincaré inequalities are sketched in Parts 7 and 8 of the Supplement.

From this it follows that $\|\mathbf{v}\|$ vanishes only if the integral of $|\mathbf{v}|^2$ vanishes and hence only if $\mathbf{v}$ vanishes, provided that $\mathbf{v}(\mathbf{x})$ is continuous. Conversely, it is immediate from the definition of $\|\mathbf{v}\|$ that it vanishes if $\mathbf{v}(\mathbf{x})$ vanishes. It follows that $\|\mathbf{v}\|$ is a positive-definite function of $\mathbf{v}(\mathbf{x})$.

Problem

Show that $\|\cdot\|$ satisfies the triangle inequality.

Thus, $\|\cdot\|$ satisfies all the conditions required to qualify as a norm

11.2 Instability

Consider a motion $\chi(\mathbf{x}, t)$ and let $\chi_e(\mathbf{x})$ be an equilibrium deformation. Let $\mathbf{u}(\mathbf{x}, t) = \chi(\mathbf{x}, t) - \chi_e(\mathbf{x})$ be the displacement from equilibrium, and consider the associated function

$$G(t) = \int_{\kappa} \rho_{\kappa}(\mathbf{x}) \, |\mathbf{u}(\mathbf{x}, t)|^2 \, dv. \tag{11.18}$$

Suppose $\mathbf{u}(\mathbf{x}, t)$ vanishes on some non-empty subset of the boundary $\partial\kappa$. Assuming $\rho_{\kappa}(\mathbf{x})$ to be continuous, we have, by the compactness of κ,

$$G(t) \leq (\max_{\mathbf{x}\in\kappa} \rho_{\kappa}) \int_{\kappa} |\mathbf{u}(\mathbf{x}, t)|^2 \, dv, \tag{11.19}$$

which implies, by eqn (11.17), that there is a positive constant, c_1 say, such that

$$G(t) \leq c_1 \|\mathbf{u}\|^2. \tag{11.20}$$

This simple result suggests a strategy whereby we seek sufficient conditions for the unbounded growth of $G(t)$. These would then ensure unbounded growth of $\|\mathbf{u}\|$ and, hence, instability of the state $\chi_e(\mathbf{x})$ as judged by the energy norm. The negation of this result would then yield necessary conditions for stability of equilibrium. Our procedure has an heuristic aspect, on which we comment later. We begin by evaluating the derivatives

$$\dot{G} = 2\int_\kappa \rho_\kappa \mathbf{u} \cdot \dot{\mathbf{u}} dv \tag{11.21}$$

and

$$\ddot{G} = 4\mathcal{K}(t) + 2\int_\kappa \rho_\kappa \mathbf{u} \cdot \ddot{\mathbf{u}} dv, \tag{11.22}$$

where $\mathcal{K}(t)$ is the kinetic energy of the body. Ignoring body forces, which play a secondary role in the argument, and assuming for simplicity that there are no local constraints, we have

$$\rho_\kappa \ddot{\mathbf{u}} = Div\hat{\mathbf{P}}(\mathbf{F};\mathbf{x}) = Div(\Delta\mathbf{P}), \tag{11.23}$$

where

$$\Delta\mathbf{P} = \hat{\mathbf{P}}(\mathbf{F};\mathbf{x}) - \mathbf{P}_e(\mathbf{x}) \tag{11.24}$$

in which $\mathbf{P}_e(\mathbf{x}) = \hat{\mathbf{P}}(\nabla\chi_e(\mathbf{x});\mathbf{x})$ is the equilibrium stress. Accordingly,

$$\ddot{G} = 4\mathcal{K}(t) + 2\int_\kappa \mathbf{u} \cdot Div(\Delta\mathbf{P}) dv. \tag{11.25}$$

At this stage, we simplify matters by confining attention to mixed displacement/dead-load problems in which $\mathbf{u}$ vanishes on a part of the boundary and the Piola traction is fixed, i.e., $(\Delta\mathbf{P})\mathbf{N}$ vanishes, on the complementary part. We integrate the second term on the right-hand side by parts using the divergence theorem, obtaining

$$\ddot{G} = 4\mathcal{K}(t) - 2\int_\kappa \Delta\mathbf{P} \cdot \Delta\mathbf{F} dv, \tag{11.26}$$

where $\Delta\mathbf{F} = \nabla\mathbf{u}$.

Next, assume there exists $\mathbf{u}(\mathbf{x},t)$ such that $\mathbf{F}(\mathbf{x},t) \in B(\mathbf{F}_e(\mathbf{x}))$, where $B(\mathbf{F}_e)$ is the open ball of radius δ centered at $\mathbf{F}_e$ defined by

$$B(\mathbf{F}_e) = \{\mathbf{F}: \quad |\nabla\mathbf{u}| < \delta\}. \tag{11.27}$$

We note that $B(\mathbf{F}_e)$ is a convex set; that is, if $\mathbf{F}_1$ and $\mathbf{F}_2$ are in $B(\mathbf{F}_e)$, then all points on the line $\mathbf{F}(u) = u\mathbf{F}_1 + (1-u)\mathbf{F}_2$, with $u \in [0,1]$, are also in $B(\mathbf{F}_e)$. To see this we use

$$\mathbf{F}(u) - \mathbf{F}_e = u(\mathbf{F}_1 - \mathbf{F}_e) + (1-u)(\mathbf{F}_2 - \mathbf{F}_e) \tag{11.28}$$

and conclude, from the triangle inequality, that

$$|\mathbf{F}(u) - \mathbf{F}_e| \leq u\,|\mathbf{F}_1 - \mathbf{F}_e| + (1-u)\,|\mathbf{F}_2 - \mathbf{F}_e| < u\delta + (1-u)\delta = \delta. \tag{11.29}$$

Thus, $\mathbf{F}(u) \in B(\mathbf{F}_e)$ and so $B(\mathbf{F}_e)$ is convex by definition. Note, however, that the domain of the constitutive function $\hat{\mathbf{P}}$, namely, the set of $\mathbf{F}$'s with $\det \mathbf{F}$ positive, is *not* convex. Therefore, our assumption on $\mathbf{u}(\mathbf{x}, t)$ is restrictive.

Problem

Prove the nonconvexity of the set $Lin^+ = \{\mathbf{F}: \quad \det \mathbf{F} > 0\}$. *Hint*: Consider

$$\mathbf{F}_1 = -3\mathbf{i} \otimes \mathbf{i} + \mathbf{j} \otimes \mathbf{j} - \mathbf{k} \otimes \mathbf{k}, \quad \mathbf{F}_2 = \mathbf{i} \otimes \mathbf{i} - 3\mathbf{j} \otimes \mathbf{j} - \mathbf{k} \otimes \mathbf{k}, \tag{11.30}$$

where $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthonormal set; both have positive determinant. Show that $\mathbf{F}(\frac{1}{2}) = -\mathbf{I}$.

If the strain–energy function is twice continuously differentiable, then Taylor's theorem with remainder (see Fleming, 1977), which is valid in any convex region and, hence, in $B(\mathbf{F}_e)$, furnishes

$$\hat{\mathbf{P}}(\mathbf{F}) = \hat{\mathbf{P}}(\mathbf{F}_e) + \mathcal{M}^*[\Delta \mathbf{F}], \tag{11.31}$$

where $\mathcal{M}^* = \mathcal{M}(\mathbf{F}_e + \alpha^* \Delta \mathbf{F})$ with $0 < \alpha^* < 1$. The same theorem yields

$$W(\mathbf{F}) = W(\mathbf{F}_e) + \hat{\mathbf{P}}(F_e) \cdot \Delta \mathbf{F} + \frac{1}{2}\mathcal{M}^{**}[\Delta \mathbf{F}] \cdot \Delta \mathbf{F}, \tag{11.32}$$

where $\mathcal{M}^{**} = \mathcal{M}(\mathbf{F}_e + \alpha^{**} \Delta \mathbf{F})$ with $0 < \alpha^{**} < 1$. Then,

$$\Delta W = \mathbf{P}_e \cdot \Delta \mathbf{F} + \frac{1}{2}\Delta \mathbf{P} \cdot \Delta \mathbf{F} + \frac{1}{2}\mathcal{L}[\Delta \mathbf{F}] \cdot \Delta \mathbf{F}, \tag{11.33}$$

where $\mathcal{L} = \mathcal{M}^{**} - \mathcal{M}^*$, and substituting into eqn (11.26) yields

$$\ddot{G} = 4\mathcal{K}(t) - 4\int_\kappa (\Delta W - \mathbf{P}_e \cdot \Delta \mathbf{F})dv - 2\int_\kappa \mathcal{L}[\Delta \mathbf{F}] \cdot \Delta \mathbf{F} dv. \tag{11.34}$$

Recalling Problem no. 2 of Section 3.2, the potential energy of a dead-loaded body is

$$\mathcal{E} = \int_\kappa W dv - \int_{\partial\kappa_p} \mathbf{p} \cdot \chi\, da, \tag{11.35}$$

where $\mathbf{p}(\mathbf{x})$ is the fixed Piola traction on the part $\partial\kappa_p$ of the boundary. This part is complementary to the part on which $\mathbf{u}$ vanishes. Conservation of energy (Problem no. 2, Section 3.2) yields

$$\frac{d}{dt}(\mathcal{E} + \mathcal{K}) = 0. \tag{11.36}$$

The change of potential energy accompanying the displacement $\mathbf{u}$ is

$$\Delta\mathcal{E} = \int_\kappa \Delta W dv - \int_{\partial\kappa_p} \mathbf{p} \cdot \mathbf{u} da, \tag{11.37}$$

where

$$\int_{\partial\kappa_p} \mathbf{p} \cdot \mathbf{u} da = \int_{\partial\kappa} \mathbf{P}_e^t \mathbf{u} \cdot \mathbf{N} da = \int_\kappa Div(\mathbf{P}_e^t \mathbf{u}) dv = \int_\kappa \mathbf{P}_e \cdot \Delta\mathbf{F} dv, \tag{11.38}$$

and we have used the fact that $Div\mathbf{P}_e$ vanishes. We have

$$\ddot{G} = 8\mathcal{K}(t) - 4\mathcal{H}(t) - 2\int_\kappa \mathcal{L}[\Delta\mathbf{F}] \cdot \Delta\mathbf{F} dv, \tag{11.39}$$

where

$$\mathcal{H}(t) = \mathcal{K}(t) + \Delta\mathcal{E}(t) \tag{11.40}$$

is the total change in mechanical energy induced by the displacement. This is fixed in time, i.e., $\mathcal{H}(t) = \mathcal{H}(0)$, by virtue of eqn (11.36).

Consider a motion with vanishing initial velocity: $\dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{0}$. Then, $\mathcal{H}(0) = \Delta\mathcal{E}_0$, the potential energy difference induced by the initial displacement $\mathbf{u}_0(\mathbf{x}) = \mathbf{u}(\mathbf{x}, 0)$. Combining, we obtain

$$\ddot{G} = 8\mathcal{K}(t) - 4\Delta\mathcal{E}_0 - 2\int_\kappa \mathcal{L}[\nabla\mathbf{u}] \cdot \nabla\mathbf{u} dv. \tag{11.41}$$

We have

$$\int_\kappa \mathcal{L}[\nabla\mathbf{u}] \cdot \nabla\mathbf{u} dv \leq \max_{\mathbf{x}\in\kappa} \mu(\mathbf{x}, t) \int_\kappa |\nabla\mathbf{u}|^2 dv, \tag{11.42}$$

where $\mu(\mathbf{x}, t)$ is the largest absolute value of the members of the set of eigenvalues of $\mathcal{L}$. This is a real number because $\mathcal{L}$ possesses major symmetry and can, therefore, be associated with a 9×9 symmetric matrix. To see this we use an argument like that used in eqn (11.12). This time let S be the symmetric 9×9 matrix consisting of the components of $\mathcal{L}$, and let s_i and λ_i be its eigenvectors and eigenvalues. Let ξ be the 9-vector consisting of the components of $\nabla\mathbf{u}$. With $\mu = \max\{|\lambda_i|\}$ we have

$$\xi' S \xi = \sum_{i=1}^{9} \lambda_i (\xi' s_i)^2 \leq \sum_{i=1}^{9} |\lambda_i| (\xi' s_i)^2 \leq \mu \sum_{i=1}^{9} (\xi' s_i)^2 = \mu |\xi|^2, \tag{11.43}$$

and, hence, eqn (11.42). Our assumption that $\mathcal{L}$ is continuous, together with $\mathbf{F} \in B(\mathbf{F}_e)$, implies that μ is bounded (see Proposition 2.18 in the book by Palis and deMelo, 1982). Accordingly, the Korn inequality yields the existence of a non-negative constant c_2, say, such that

$$\int_{\kappa} \mathcal{L}[\nabla \mathbf{u}] \cdot \nabla \mathbf{u} dv \leq c_2 \|\mathbf{u}\|^2. \tag{11.44}$$

Because $\mathcal{K}(t) \geq 0$, we conclude that

$$\ddot{G} \geq -4\Delta\mathcal{E}_0 + R, \quad \text{with} \quad R = O(\|\mathbf{u}\|^2). \tag{11.45}$$

We are finally ready to establish our main result. Suppose the equilibrium state is *stable* in the sense that $\|\mathbf{u}\| < \varepsilon$ for some assigned ε, for all $t \geq 0$. Let $\mathbf{u}_0(\mathbf{x})$ be such that $\Delta\mathcal{E}_0 < 0$, i.e., $\mathcal{E}[\chi_e + \mathbf{u}_0] < \mathcal{E}[\chi_e]$, and suppose that $\chi_e(\mathbf{x}) + \mathbf{u}_0(\mathbf{x})$ is *not* equilibrated, i.e., $\ddot{G}(0) \neq 0$ (cf. eqn (11.22)). Then, we can choose ε small enough to ensure that

$$-4\Delta\mathcal{E}_0 + R \geq A, \tag{11.46}$$

a positive constant. It follows that $\ddot{G}(t) \geq A$ and, hence, from eqns (11.20) and (11.45), that

$$\|\mathbf{u}\|^2 \geq \frac{1}{2} c_3 t^2 + c_4 \tag{11.47}$$

for some positive constant c_3 and some constant c_4. We have used the fact that $\dot{G}(0) = 0$ (cf. eqn (11.21)). Thus there is a time, t_1 say, such that $\|\mathbf{u}(\mathbf{x}, t_1)\| \geq \varepsilon$, contrary to the stability hypothesis. We have thus shown that if $\chi_0(\mathbf{x}) = \chi_e(\mathbf{x}) + \mathbf{u}_0(\mathbf{x})$ is a non-equilibrium, kinematically admissible displacement field ($\mathbf{u}_0 = \mathbf{0}$ on the complement of $\partial\kappa_p$), then for the *equilibrium* configuration $\chi_e(\mathbf{x})$ to be unstable with respect to the energy norm, it is sufficient that $\mathbf{u}_0(\mathbf{x})$ be such as to furnish $\Delta\mathcal{E}_0 < 0$.

The negation of this statement furnishes a *necessary* condition for stability: If $\chi_e(\mathbf{x})$ is stable with respect to the energy norm, then it is necessary that

$$\mathcal{E}[\chi_e] \leq \mathcal{E}[\chi_0] \tag{11.48}$$

for *all* non-equilibrium deformations $\chi_0(\mathbf{x})$ that meet assigned position data on the complement of $\partial\kappa_p$. This is the famous *energy criterion* of elastic stability theory. It effectively reduces the equilibrium problem to the central problem of the Calculus of Variations: Find a vector-valued function that minimizes a scalar-valued *functional*. Here, of course, the potential energy. Needless to say, the criterion is meaningful only for conservative problems.

The energy criterion furnishes the point of departure for a vast amount of modern research on finite elasticity. Indeed, the question of the existence of energy minimizers was settled in the landmark paper by J. Ball (1977). This constitutes a major milestone in 20th century research on Finite Elasticity and the Calculus of Variations more broadly.

Of course, we have made a number of questionable assumptions in the course of deriving the energy criterion, not least among these being the existence of a motion having gradient in the ball $B(\mathbf{F}_\epsilon(\mathbf{x}))$ at each $\mathbf{x} \in \kappa$. Beyond this, we have assumed this motion to be sufficiently smooth in space and time as to justify our various formal manipulations. To this day, conditions ensuring such global regularity are not known. Nevertheless, our development offers a formal justification of the energy criterion and thus provides a degree of confidence in its validity. The work of Koiter (1966), in particular, offers arguments in support of this criterion as being both necessary and sufficient for stability.

Problem

Consider the problem of the extension of a unit cube of Mooney-Rivlin material under equibiaxial force f (cf. Problem no. 6(b) in Section 7.2). (a) Show that the potential energy of the deformed material is

$$E(\lambda_1, \lambda_2) = \omega(\lambda_1, \lambda_2) - f\lambda_1 - f\lambda_2, \tag{11.49}$$

where $\omega(\lambda_1, \lambda_2)$ is the three–dimensional strain energy obtained by imposing the incompressibility condition $\lambda_1\lambda_2\lambda_3 = 1$. (b) Show that equilibrium corresponds to stationarity of the energy; i.e., $\partial E/\partial\lambda_\alpha = 0$ for $\alpha = 1, 2$. (c) Let λ^* be the critical value of equibiaxial stretch at which bifurcation from equibiaxial to unequal biaxial stretch is possible. Show that the solution with equibiaxial stretch ($\lambda_1 = \lambda_2 = \lambda$, say) is stable if $\lambda < \lambda^*$ and unstable if $\lambda > \lambda^*$. Show that the solution with unequal biaxial stretch is stable. Thus, in practice the cube undergoes a transition from equibiaxial to unequal biaxial stretch at $\lambda > \lambda^*$.

Hint: The energy is minimized at an equilibrium state if and only if the matrix $\partial^2 E/\partial\lambda_\alpha\partial\lambda_\beta$ is positive definite. Thus, failure of positive definiteness marks the transition from stability to instability.

Remark By restricting attention to deformations that are homogeneous, we have provided only a partial analysis of stability, because we have not allowed non-homogeneous deformations into the competition for the minimum of the energy. In other words, it is conceivable that a non-homogeneous deformation might be able to produce a lower-energy state than the equilibrium states considered here.

11.3 Quasiconvexity

We inquire into some implications of the energy criterion eqn (11.48). Staying with mixed displacement/dead-load problems for illustrative purposes, this criterion may be written in the form

$$\int_{\kappa}[W(\mathbf{F}_e+\Delta\mathbf{F})-W(\mathbf{F}_e)]dv-\int_{\partial\kappa_p}\mathbf{p}\cdot\Delta\chi\, da\geq 0, \tag{11.50}$$

for all $\Delta\chi$ that vanish on the complement of $\partial\kappa_p$. Among these, we consider

$$\Delta\chi(\mathbf{x})=\varepsilon\phi(\mathbf{z});\quad \mathbf{z}=\varepsilon^{-1}(\mathbf{x}-\mathbf{x}_0) \tag{11.51}$$

with ε a positive constant, $\mathbf{x}_0$ an interior point of κ and $\phi\neq\mathbf{0}$ only in some region D contained in the interior of κ. Note that the the nature of the boundary load potential is irrelevant for this choice of $\Delta\chi$ and hence that our further considerations are valid in more general cases of (conservative) loading.

We have $\Delta\mathbf{F}=\nabla\phi(\mathbf{z})$ in this case, where the gradient is taken with respect to $\mathbf{z}$. Accordingly, eqn (11.50) reduces to

$$\int_{\kappa'}[W(\mathbf{F}_e(\mathbf{x})+\nabla\phi(\mathbf{z});\mathbf{x})-W(\mathbf{F}_e(\mathbf{x});\mathbf{x})]dv(\mathbf{x})\geq 0, \tag{11.52}$$

where κ' is the image of D under the inverse of the map eqn $(11.51)_2$. Note that the map defined by eqn $(11.51)_2$ has gradient $\varepsilon^{-1}\mathbf{I}$, with determinant ε^{-3}. Then, $dv(\mathbf{z})=\varepsilon^{-3}dv(\mathbf{x})$ and, after division by ε^3, we derive

$$\int_D[W(\mathbf{F}_e(\mathbf{x}_0+\varepsilon\mathbf{z})+\nabla\phi(\mathbf{z});\mathbf{x}_0+\varepsilon\mathbf{z})-W(\mathbf{F}_e(\mathbf{x}_0+\varepsilon\mathbf{z});\mathbf{x}_0+\varepsilon\mathbf{z})]dv(\mathbf{z})\geq 0. \tag{11.53}$$

We now let $\varepsilon\to 0$ and use the dominated convergence theorem (see Fleming, 1977) to conclude that

$$\int_D[W(\mathbf{F}_e(\mathbf{x}_0)+\nabla\phi(\mathbf{z});\mathbf{x}_0)-W(\mathbf{F}_e(\mathbf{x}_0);\ \mathbf{x}_0)]dv(\mathbf{z})\geq 0 \tag{11.54}$$

for all $\mathbf{x}_0$ and for any ϕ with $\phi=\mathbf{0}$ on ∂D. This is the *quasiconvexity* condition. It was discovered by Morrey, and plays a major role in Ball's existence theorem for energy minimizers. A particularly good account may be found in Ciarlet (1988).

We have derived quasiconvexity as a necessary condition for energy minimization. It has an interesting physical interpretation: Consider a uniform material with strain energy $W^*(\mathbf{F})=W(\mathbf{F};\ \mathbf{x}_0)$. Then, quasiconvexity means that the energy

$$F[\phi]=\int_D W^*(\mathbf{F}_0+\nabla\phi)dv,\quad \text{with}\quad \phi=\mathbf{0}\quad\text{on}\quad \partial D\quad\text{and}\quad \mathbf{F}_0=\mathbf{F}_e(\mathbf{x}_0), \tag{11.55}$$

is minimized absolutely by $\phi\equiv\mathbf{0}$, i.e., by the *homogeneous* deformation $\mathbf{F}_0\mathbf{z}$. This is precisely the potential energy for a pure displacement boundary-value problem in which perturbations of the deformation vanish on the entire boundary.

Note that the quasiconvexity condition imposes a restriction on the deformation $\mathbf{F}_e(\mathbf{x}_0)$, for each $\mathbf{x}_0\in\kappa$; if there are any points where it is violated, for some ϕ, then $\chi_e(\mathbf{x})$ cannot be an energy minimizing deformation. However, the quasiconvexity condition is manifestly non-local and this poses an obstacle to its direct verification. This fact has given impetus to the search for purely local conditions that imply quasiconvexity.

11.4 Ordinary convexity

One of these local conditions is ordinary convexity, i.e.,

$$W(\mathbf{F}_0 + \nabla\phi;\ \mathbf{x}_0) - W(\mathbf{F}_0;\ \mathbf{x}_0) \geq W_{\mathbf{F}}(\mathbf{F}_0;\ \mathbf{x}_0) \cdot \nabla\phi. \tag{11.56}$$

We confine attention to unconstrained materials. Then,

$$\begin{aligned}\int_D [W(\mathbf{F}_0 + \nabla\phi;\ \mathbf{x}_0) - W(\mathbf{F}_0;\ \mathbf{x}_0)]dv &\geq \hat{\mathbf{P}}(\mathbf{F}_0;\ \mathbf{x}_0) \cdot \int_D \nabla\phi(\mathbf{z})dv \\ &= \hat{\mathbf{P}}(\mathbf{F}_0;\ \mathbf{x}_0) \cdot \int_{\partial D} \phi \otimes \mathbf{N}da,\end{aligned} \tag{11.57}$$

and this vanishes if ϕ vanishes on ∂D. Accordingly, convexity is sufficient for quasiconvexity. However, this condition suffers from major drawbacks and, therefore, cannot be regarded as realistic. We elaborate on two of these here.

11.4.1 Objections to ordinary convexity

(a) Consistency with the symmetry of the Cauchy stress implies that compressive states of stress are impossible.

To see this we write eqn (11.56) in the form

$$W(\tilde{\mathbf{F}}) - W(\mathbf{F}) \geq \hat{\mathbf{P}}(\mathbf{F}) \cdot (\tilde{\mathbf{F}} - \mathbf{F}), \tag{11.58}$$

which purports to hold for all $\mathbf{F}$ and $\tilde{\mathbf{F}}$ with positive determinant. Therefore, it holds if $\tilde{\mathbf{F}} = \mathbf{Q}\mathbf{F}$ with $\mathbf{Q}$ a rotation. We have seen that $W(\tilde{\mathbf{F}}) = W(\mathbf{F})$ in this case if and only if the Cauchy stress $\mathbf{T}$ is symmetric. Thus,

$$\begin{aligned}0 \geq \hat{\mathbf{P}}(\mathbf{F}) \cdot [(\mathbf{Q} - \mathbf{I})\mathbf{F}] &= tr\{\hat{\mathbf{P}}[(\mathbf{Q} - \mathbf{I})\mathbf{F}]^t\} \\ &= tr[\hat{\mathbf{P}}\mathbf{F}^t(\mathbf{Q} - \mathbf{I})^t] = J\mathbf{T} \cdot (\mathbf{Q} - \mathbf{I}),\end{aligned} \tag{11.59}$$

where $\mathbf{T}$ is the Cauchy stress associated with $\mathbf{F}$. Consider a one-parameter family of rotations $\mathbf{Q}(\varepsilon)$ with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $\mathbf{Q}(0) = \mathbf{I}$, and define $f(\varepsilon) = \mathbf{T} \cdot [\mathbf{Q}(\varepsilon) - \mathbf{I}]$. Then, $f(0) = 0, f'(\varepsilon) = \mathbf{T} \cdot \mathbf{Q}'(\varepsilon)$ and $f''(\varepsilon) = \mathbf{T} \cdot \mathbf{Q}''(\varepsilon)$. Note that $\mathbf{\Omega}(\varepsilon) \equiv \mathbf{Q}'(\varepsilon)\mathbf{Q}(\varepsilon)^t$ is skew symmetric. We have $\mathbf{Q}'(\varepsilon) = \mathbf{\Omega}(\varepsilon)\mathbf{Q}(\varepsilon)$ and

$$\mathbf{Q}''(\varepsilon) = \mathbf{\Omega}'(\varepsilon)\mathbf{Q}(\varepsilon) + \mathbf{\Omega}(\varepsilon)^2\mathbf{Q}(\varepsilon), \tag{11.60}$$

where $\boldsymbol{\Omega}'(\varepsilon)$ is also skew symmetric, as are $\omega \equiv \boldsymbol{\Omega}(0)$ and $\alpha \equiv \boldsymbol{\Omega}'(0)$. Accordingly, $\mathbf{Q}'(0) = \omega$ and $\mathbf{Q}''(0) = \alpha + \omega^2$. These imply that $f'(0) = 0$ and $f''(0) = \mathbf{T} \cdot \omega^2$, by virtue of the symmetry of $\mathbf{T}$.

Inequality eqn (11.59) then implies, for small ε, that

$$0 \geq f(\varepsilon) = \frac{1}{2}\varepsilon^2[f''(0) + o(\varepsilon^2)/\varepsilon^2]. \tag{11.61}$$

Dividing by ε^2 and passing to the limit, we conclude that $f''(0) \leq 0$, i.e.,

$$\mathbf{T} \cdot \omega^2 \leq 0, \tag{11.62}$$

for all skew ω.

Problems

1. Show that eqn (11.62) is equivalent to

$$\omega_{12}^2(t_1 + t_2) + \omega_{13}^2(t_1 + t_3) + \omega_{23}^2(t_2 + t_3) \geq 0, \tag{11.63}$$

where t_i are the principal Cauchy stresses and ω_{ij} are the components of ω on the principal stress axes. Therefore, convexity and the symmetry of the Cauchy stress together imply that the state of stress is tensile in the sense that

$$t_1 + t_2 \geq 0, \quad t_1 + t_3 \geq 0 \quad \text{and} \quad t_2 + t_3 \geq 0 \tag{11.64}$$

pointwise, at all deformations. This is an unrealistically severe restriction to impose on a general boundary-value problem.

2. Consider a homogeneous, compressible elastic material (W not explicitly dependent on $\mathbf{x}$) subjected to a prescribed dead-load traction distribution $\mathbf{p}(\mathbf{x})$ on its *entire* boundary.
   (i) Show that, in the absence of body forces, a necessary condition for the existence of an equilibrium deformation $\chi(\mathbf{x})$ is that $\int_{\partial\kappa} \mathbf{p}da = \mathbf{0}$.
   (ii) Suppose the equilibrium deformation χ is uniform in the sense that its gradient $\mathbf{F}$ is independent of $\mathbf{x}$. As we have shown, the change in potential energy associated with a kinematically admissible deformation $\chi \to \bar{\chi}$ is

$$\Delta\mathcal{E} = \int_\kappa [W(\nabla\bar{\chi}) - W(\mathbf{F}) - \hat{\mathbf{P}}(\mathbf{F}) \cdot (\nabla\bar{\chi} - \mathbf{F})]dv. \tag{11.65}$$

   Show that $\Delta\mathcal{E} \geq 0$ for *arbitrary* kinematically admissible $\bar{\chi}$ *if, and only if*, the integrand is non-negative at every point of the body. Thus, conclude that in this case, contrary to the situation in mixed position/traction problems, $\mathbf{F}$ must satisfy the condition of ordinary convexity.

(iii) Use this result to prove that a *homogeneously* deformed bar in equilibrium under equal and opposite compressive tractions at its ends (and zero tractions on the remainder of its boundary) is unstable.

(b) Strict convexity precludes buckling in the mixed dead-load problem.

Using $\mathbf{p} = \mathbf{P}_e\mathbf{N}$ on $\partial\kappa_p$ and $\Delta\chi = 0$ on the complement of $\partial\kappa_p$, we can write

$$\begin{aligned}
&\int_\kappa [W(\mathbf{F}_e + \Delta\mathbf{F}) - W(\mathbf{F}_e)]dv - \int_{\partial\kappa_p} \mathbf{p}\cdot\Delta\chi\, da \\
&= \int_\kappa [W(\mathbf{F}_e + \Delta\mathbf{F}) - W(\mathbf{F}_e)]dv - \int_{\partial\kappa} \mathbf{P}_e^t\Delta\chi\cdot\mathbf{N}da \\
&= \int_\kappa [W(\mathbf{F}_e + \Delta\mathbf{F}) - W(\mathbf{F}_e) - \hat{\mathbf{P}}(\mathbf{F}_e)\cdot\Delta\mathbf{F}]dv,
\end{aligned} \tag{11.66}$$

where we have invoked $Div\mathbf{P}_e = \mathbf{0}$ in the course of integrating by parts. From this is it obvious that strict convexity; i.e., strict inequality in eqn (11.56), implies that $\Delta\mathcal{E} > 0$ for any non-zero $\Delta\chi$ that vanishes on the complement of $\partial\kappa_p$. Therefore, strict convexity is sufficient for an equilibrium state to furnish a strict minimum of the potential energy.

Let $\mathbf{F}_1 = \mathbf{F}_e$ and $\mathbf{F}_2 = \mathbf{F}_1 + \Delta\mathbf{F}$ be the gradients of two *equilibrium* deformations χ_1 and χ_2, respectively, and suppose χ_1 is a strict minimizer. Then,

$$\int_\kappa [W(\mathbf{F}_2) - W(\mathbf{F}_1) - \hat{\mathbf{P}}(\mathbf{F}_1)\cdot(\mathbf{F}_2 - \mathbf{F}_1)]dv > 0, \tag{11.67}$$

provided that $\mathbf{F}_2 \neq \mathbf{F}_1$. In the same way, if χ_2 is a strict minimizer, then

$$\int_\kappa [W(\mathbf{F}_1) - W(\mathbf{F}_2) - \hat{\mathbf{P}}(\mathbf{F}_2)\cdot(\mathbf{F}_1 - \mathbf{F}_2)]dv > 0, \tag{11.68}$$

again, provided that $\mathbf{F}_2 \neq \mathbf{F}_1$. We re-write this as

$$-\int_\kappa [W(\mathbf{F}_2) - W(\mathbf{F}_1) - \hat{\mathbf{P}}(\mathbf{F}_2)\cdot(\mathbf{F}_2 - \mathbf{F}_1)]dv > 0 \tag{11.69}$$

and add it to (11.67), concluding that

$$\int_\kappa [\hat{\mathbf{P}}(\mathbf{F}_2) - \hat{\mathbf{P}}(\mathbf{F}_1)\cdot(\mathbf{F}_2 - \mathbf{F}_1)]dv > 0, \quad \text{provided that} \quad \mathbf{F}_2 \neq \mathbf{F}_1. \tag{11.70}$$

However, since $Div(\Delta\mathbf{P}) = \mathbf{0}$ in κ and $(\Delta\mathbf{P})\mathbf{N} = \mathbf{0}$ on $\partial\kappa_p$, where $\Delta\mathbf{P} = \hat{\mathbf{P}}(\mathbf{F}_2) - \hat{\mathbf{P}}(\mathbf{F}_1)$; and, since $\Delta\chi(= \chi_2 - \chi_1)$ vanishes on the complement of $\partial\kappa_p$, we have

$$0 = \int_{\partial\kappa} (\Delta\mathbf{P})'\Delta\chi \cdot \mathbf{N}da = \int_{\kappa} \Delta\mathbf{P} \cdot \Delta\mathbf{F}dv; \tag{11.71}$$

that is,

$$\int_{\kappa} [\hat{\mathbf{P}}(\mathbf{F}_2) - \hat{\mathbf{P}}(\mathbf{F}_1) \cdot (\mathbf{F}_2 - \mathbf{F}_1)]dv = 0, \tag{11.72}$$

which is reconciled with eqn (11.70) if and only if $\mathbf{F}_2 = \mathbf{F}_1$. Then, $\chi_2 - \chi_1 = \mathbf{c}$, a rigid translation of the entire body. If the complement of $\partial\kappa_p$ is non-empty, $\mathbf{c}$ vanishes and the equilibrium deformations coincide. We conclude that if an equilibrium deformation minimizes the energy strictly, then it is unique. In particular, then, strict convexity of the energy implies unqualified uniqueness of equilibria in the mixed position/dead-load problem. Thus, strict convexity precludes buckling—the phenomenon of bifurcation of equilibrium—under all dead loads, and is, therefore, unrealistic.

Nevertheless, the negation of this result furnishes the well known Euler criterion for potential instability: Non-uniqueness of equilibria (e.g., buckling) implies that they are not strict minimizers of the energy, i.e., that $\Delta\mathcal{E} \leq 0$ relative to the considered equilibrium deformation, for some $\Delta\chi$ that vanishes on the complement of $\partial\kappa_p$. Accordingly, such equilibria fail to satisfy the strict form of the necessary condition eqn (11.48) for stability and might, therefore, be unstable. Said differently, the onset of non-uniqueness of equilibria signals a potential instability. We cannot assert definitively that non-uniqueness implies instability of equilibrium because the strict form of eqn (11.48) is not known to be necessary for stability. The Euler criterion is studied in more detail in Chapter 12.

11.5 Polyconvexity

This is the statement that there exists a function $\Phi(\mathbf{F}, \mathbf{F}^*, \det\mathbf{F})$, jointly convex in its arguments, such that $W(\mathbf{F}) = \Phi(\mathbf{F}, \mathbf{F}^*, \det\mathbf{F})$, i.e.,

$$W(\mathbf{F}) - W(\mathbf{F}_0) \geq \mathbf{A}(\mathbf{F}_0) \cdot (\mathbf{F} - \mathbf{F}_0) + \mathbf{B}(\mathbf{F}_0) \cdot (\mathbf{F}^* - \mathbf{F}_0^*) + C(\mathbf{F}_0)(\det\mathbf{F} - \det\mathbf{F}_0), \tag{11.73}$$

with

$$\mathbf{A}(\mathbf{F}) = \Phi_{\mathbf{F}}, \quad \mathbf{B}(\mathbf{F}) = \Phi_{\mathbf{F}^*} \quad \text{and} \quad C(\mathbf{F}) = \Phi_{\det\mathbf{F}}. \tag{11.74}$$

Consider a deformation with gradient $\mathbf{F} = \mathbf{F}_0 + \nabla\phi$, where $\mathbf{F}_0$ is uniform. Then,

$$\begin{aligned}\int_D [W(\mathbf{F}_0 + \nabla\phi) - W(\mathbf{F}_0)]dv \\ \geq \mathbf{A}(\mathbf{F}_0) \cdot \int_D \nabla\phi dv + \mathbf{B}(\mathbf{F}_0) \cdot \int_D (\mathbf{F}^* - \mathbf{F}_0^*)dv + C(\mathbf{F}_0)\int_D (\det\mathbf{F} - \det\mathbf{F}_0)dv.\end{aligned} \tag{11.75}$$

We assign $\phi = \mathbf{0}$ on ∂D. We show that the right-hand side vanishes, and thus that polyconvexity is sufficient for quasiconvexity. To this end we derive three identities:

(a) $\int_D \nabla\phi dv = \int_{\partial D} \phi \otimes \mathbf{N} dv$. This of course is a variant of the divergence theorem. Obviously, the integral vanishes because ϕ vanishes on the boundary.

(b) Recall that $F^*_{iA} = \psi_{iAB,B}$ with $\psi_{iAB} = \frac{1}{2}e_{ijk}e_{ABC}\chi_j\chi_{k,C} = -\psi_{iBA}$, where e is the permutation symbol ($e_{123} = +1$, etc.). Thus, by the divergence theorem,

$$\int_D (\mathbf{F}^* - \mathbf{F}^*_0)dv = \mathbf{e}_i \otimes \mathbf{E}_A \int_{\partial D} (\psi_{iAB} - \psi^0_{iAB})N_B da. \tag{11.76}$$

Let $v(\mathbf{x})$ be a scalar field and consider the vector $e_{ABC}v_{,C}N_B = (\mathbf{N} \times \nabla v)_A$. We have $\mathbf{N} \times \nabla v = \mathbf{N} \times \mathbb{P}(\nabla v)$ on ∂D, where $\mathbb{P} = \mathbf{I} - \mathbf{N} \otimes \mathbf{N}$ is the projection onto the local tangent plane of ∂D. Thus, $\mathbf{N} \times \nabla v$ involves only the tangential derivatives of v in the surface, which in turn are determined by the values of v on the surface. Choosing $v = \chi_{k|\partial D} = \chi^0_{k|\partial D}$, we find that $\psi_{iAB}N_B = \psi^0_{iAB}N_B$ on ∂D because ϕ vanishes there. Thus, $\int_D (\mathbf{F}^* - \mathbf{F}^*_0)dv$ vanishes.

(c) It is elementary to show that $\det \mathbf{F} = \frac{1}{6}e_{ijk}e_{ABC}F_{iA}F_{jB}F_{kC} = \frac{1}{3}F_{iA}F^*_{iA} = \frac{1}{3}(\chi_i F^*_{iA})_{,A}$, a divergence. We have, of course, invoked the Piola identity. Then,

$$\int_D \det \mathbf{F} dv = \frac{1}{3}\int_{\partial D} \chi \cdot \mathbf{F}^*\mathbf{N} da = \frac{1}{3}\int_{\chi(\partial D)} \chi \cdot \mathbf{n} da, \tag{11.77}$$

where $\chi(\partial D)$ is the image of ∂D under the deformation map and we have used Nanson's formula. Because $\phi = \mathbf{0}$ on ∂D the same result follows on replacing $\mathbf{F}$ by $\mathbf{F}_0$. Accordingly, $\int_D (\det \mathbf{F} - \det \mathbf{F}_0)dv$ vanishes.

We have shown that polyconvexity implies quasiconvexity. Indeed, polyconvexity is central to the hypotheses underpinning Ball's existence theorem. It does not suffer from the drawbacks of ordinary convexity. For example, Ball (1977) has shown that Ogden's strain-energy function satisfies polyconvexity. Further examples of polyconvex strain energies are discussed in the papers by Steigmann (2003a,b).

Thus far, we have shown that both ordinary convexity and polyconvexity imply quasiconvexity. However, polyconvexity does not imply convexity, as shown by the following counter-example. In general, we have

$$W(\mathbf{F} + \Delta\mathbf{F}) - W(\mathbf{F}) - \hat{\mathbf{P}}(\mathbf{F}) \cdot \Delta\mathbf{F} = \frac{1}{2}\mathcal{M}(\mathbf{F})[\Delta\mathbf{F}] \cdot \Delta\mathbf{F} + o(|\Delta\mathbf{F}|^2). \tag{11.78}$$

Set $\Delta\mathbf{F} = \theta\mathbf{A}$ with $\mathbf{A}$ fixed. Then if $W(\mathbf{F})$ is convex,

$$\frac{1}{2}\theta^2\{\mathcal{M}(\mathbf{F})[\mathbf{A}] \cdot \mathbf{A} + o(\theta^2)/\theta^2\} \geq 0. \tag{11.79}$$

Dividing by θ^2 and passing to the limit, we conclude that $W(\mathbf{F})$ is convex only if

$$\mathcal{M}(\mathbf{F})[\mathbf{A}] \cdot \mathbf{A} \geq 0 \tag{11.80}$$

for all $\mathbf{A}$; in other words, only if the function $G(\theta) = \frac{1}{2}\theta^2\mathcal{M}(\mathbf{F})[\mathbf{A}] \cdot \mathbf{A}$ satisfies $G''(\theta) \geq 0$.

Consider the deformation gradient

$$\mathbf{F}(\theta) = \theta(2\mathbf{i} \otimes \mathbf{i} + \mathbf{j} \otimes \mathbf{j} + \mathbf{k} \otimes \mathbf{k}) + (1 - \theta)(\mathbf{i} \otimes \mathbf{i} + 2\mathbf{j} \otimes \mathbf{j} + \mathbf{k} \otimes \mathbf{k}) \tag{11.81}$$

with $\theta \in [0, 1]$. This has $\det \mathbf{F}(\theta) = (1 + \theta)(2 - \theta) > 0$, and is, therefore, an admissible deformation gradient.

Consider $W(\mathbf{F}) = \det \mathbf{F}$, which is trivially polyconvex. Picking $\theta\mathbf{A} = \mathbf{F}(\theta) - \mathbf{F}(0)$, we find that $G(\theta) = -\frac{1}{2}\theta^2$. Then, $G''(\theta) = -1$, proving that $\det \mathbf{F}$ is not a convex function of $\mathbf{F}$.

11.6 Rank-one convexity

Consider the region $D \subset \kappa$ in the definition of quasiconvexity. We follow a construction due to Graves (1939), but confine attention to two dimensions for the sake of simplicity. Graves' treatment is valid in n dimensions. Let the origin of $\mathbf{z}$ be located at $\mathbf{x}_0$, and attach an orthonormal basis $\{\mathbf{M}, \mathbf{N}\}$ there. Consider a lens-shaped region $R \subset D$ as shown in Figure 11.1, and let $x = \mathbf{M} \cdot \mathbf{z}$ and $y = \mathbf{N} \cdot \mathbf{z}$.

Then,

$$R = \{(x, y) : |x| < h, \quad 0 < y < B(x)\} = R_1 \cup R_2, \tag{11.82}$$

where

$$R_1 = \{(x, y) : |x| < h, \quad 0 < y < \theta B(x)\}, \quad R_2 = R \setminus R_1. \tag{11.83}$$

with $0 < \theta < 1$ and $B(x) = h^2 - x^2$. Consider the function

$$F(\mathbf{z}) = \begin{matrix} (1-\theta)y & \text{in} & R_1 \\ -\theta[y - B(x)] & \text{in} & R_2 \\ 0, & \text{outside} & R. \end{matrix} \tag{11.84}$$

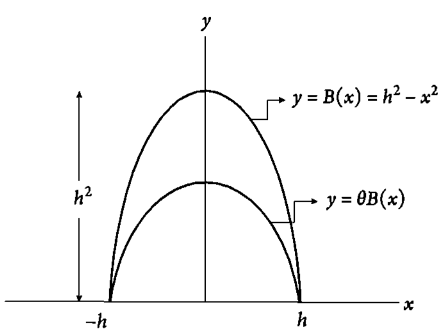

Figure 11.1 Lens-shaped region

Note that

$$F(\mathbf{z}) = \begin{matrix} \theta(1-\theta)B & \text{at} & y=\theta B \\ -\theta B(\theta-1) & \text{at} & y=\theta B' \end{matrix} \tag{11.85}$$

so that F is continuous.

We define

$$\phi(\mathbf{z}) = \mathbf{a}F(\mathbf{z}), \tag{11.86}$$

with $\mathbf{a}$ fixed. Its gradient is

$$\nabla\phi = \mathbf{a} \otimes \nabla F, \tag{11.87}$$

where

$$\begin{aligned} \nabla F &= F_x \nabla x + F_y \nabla y = F_x \mathbf{M} + F_y \mathbf{N} \\ &= \begin{matrix} (1-\theta)\mathbf{N} & \text{in} & R_1 \\ -\theta\mathbf{N} - \theta(2x)\mathbf{M} & \text{in} & R_2 \end{matrix} \rightarrow \begin{matrix} (1-\theta)\mathbf{N} & \text{in} & R_1 \\ -\theta\mathbf{N} & \text{in} & R_2 \end{matrix}, \end{aligned} \tag{11.88}$$

as $h \to 0$. We also have $F \to 0$ as $h \to 0$. Thus,

$$\phi(\mathbf{z}) \to \mathbf{0}, \quad \nabla\phi \to \begin{matrix} (1-\theta)\mathbf{a}\otimes\mathbf{N} & \text{in} & R_1 \\ -\theta\mathbf{a}\otimes\mathbf{N} & \text{in} & R_2 \end{matrix}. \tag{11.89}$$

Let

$$\begin{aligned} V &= measR = \int_{-h}^{h} B(x)dx, \\ V_1 &= measR_1 = \int_{-h}^{h} \theta B(x)dx = \theta V, \\ V_2 &= measR_2 = (1-\theta)V. \end{aligned} \tag{11.90}$$

Then, the quasiconvexity condition yields

$$\begin{aligned} 0 &\le V^{-1} \int_R [W(\mathbf{F}_0 + \nabla\phi) - W(\mathbf{F}_0)]dv \\ &= \frac{\theta}{V_1} \int_{R_1} W(\mathbf{F}_0 + \nabla\phi)dv + \frac{1-\theta}{V_2} \int_{R_2} W(\mathbf{F}_0 + \nabla\phi)dv - \frac{1}{V} \int_R W(\mathbf{F}_0)dv. \end{aligned} \tag{11.91}$$

Using the mean-value theorem in each integral, letting $h \to 0$ and dividing by θ, we obtain the pointwise condition

$$W[\mathbf{F}_0 + (1-\theta)\mathbf{a}\otimes\mathbf{N}] + \frac{1-\theta}{\theta} W(\mathbf{F}_0 - \theta\mathbf{a}\otimes\mathbf{N}) - \frac{1}{\theta} W(\mathbf{F}_0) \ge 0; \quad 0 < \theta < 1. \tag{11.92}$$

For small θ,

$$W(\mathbf{F}_0 - \theta\mathbf{a} \otimes \mathbf{N}) = W(\mathbf{F}_0) - \theta\hat{\mathbf{P}}(\mathbf{F}_0) \cdot \mathbf{a} \otimes \mathbf{N} + o(\theta), \tag{11.93}$$

and, therefore,

$$\frac{1-\theta}{\theta} W(\mathbf{F}_0 - \theta\mathbf{a} \otimes \mathbf{N}) = \frac{1-\theta}{\theta} W(\mathbf{F}_0) - (1-\theta)\hat{\mathbf{P}}(\mathbf{F}_0) \cdot \mathbf{a} \otimes \mathbf{N} + (1-\theta)\frac{o(\theta)}{\theta}. \tag{11.94}$$

It follows from eqn (11.92) that

$$W[\mathbf{F}_0 + (1-\theta)\mathbf{a} \otimes \mathbf{N}] - W(\mathbf{F}_0) - (1-\theta)\hat{\mathbf{P}}(\mathbf{F}_0) \cdot \mathbf{a} \otimes \mathbf{N} + (1-\theta)\frac{o(\theta)}{\theta} \geq 0, \tag{11.95}$$

and passage to the limit yields

$$W(\mathbf{F}_0 + \mathbf{a} \otimes \mathbf{N}) - W(\mathbf{F}_0) - \hat{\mathbf{P}}(\mathbf{F}_0) \cdot \mathbf{a} \otimes \mathbf{N} \geq 0 \tag{11.96}$$

in which $\mathbf{a} \otimes \mathbf{N}$ is arbitrary.

This is the condition of *rank-one convexity*, so named because it requires the convexity of W with respect to rank-one perturbations of the deformation gradient. We have derived it as a consequence of quasiconvexity. Accordingly, it is *necessary* for quasiconvexity, and therefore necessary for minimum energy. In principle, it constitutes a restriction on the value of the deformation gradient $\mathbf{F}_0 = \mathbf{F}_\epsilon(\mathbf{x}_0)$, at each $\mathbf{x}_0 \in \kappa$. If there is any $\mathbf{a} \otimes \mathbf{N}$ such that inequality eqn (11.96) is violated at some $\mathbf{x}_0$, then the field $\chi_\epsilon(\mathbf{x})$ cannot be an energy minimizer and, hence, cannot be stable. In particular, eqn (11.96) does not constitute a restriction on the *function* W. That is, it could be violated at some deformation gradients in the domain of the strain–energy function. Such gradients must then be relegated to sets of measure zero in κ if the deformation is to minimize the potential energy.

In the case of incompressibility eqn (11.96) remains valid with the restriction $\mathbf{a} \cdot \mathbf{F}_0^*\mathbf{N} = 0$. The derivation in this case may be found in the paper by Fosdick and MacSithigh (1986).

Problem

Show that $\det(\mathbf{F} + \mathbf{a} \otimes \mathbf{N}) = \det \mathbf{F} + \mathbf{a} \cdot \mathbf{F}^*\mathbf{N}$.

As $\mathbf{N}$ is a unit vector we have

$$W(\mathbf{F}_0 + \mathbf{a} \otimes \mathbf{N}) - W(\mathbf{F}_0) - \hat{\mathbf{P}}(\mathbf{F}_0) \cdot \mathbf{a} \otimes \mathbf{N} = \frac{1}{2}\mathcal{M}(\mathbf{F}_0)[\mathbf{a} \otimes \mathbf{N}] \cdot \mathbf{a} \otimes \mathbf{N} + o(|\mathbf{a}|^2). \tag{11.97}$$

Dividing eqn (11.96) through by $|\mathbf{a}|^2$ and passing to the limit yields the *Legendre–Hadamard inequality*

$$\mathbf{a} \cdot \mathbf{A}(\mathbf{F}_0, \mathbf{N})\mathbf{a} \geq 0, \tag{11.98}$$

where $\mathbf{A}(\mathbf{F}_0; \mathbf{N})$ is the acoustic tensor defined by eqn (9.22). If the deformation is such that Legendre–Hadamard condition is violated at any point in the body, then it is not an energy minimizer. The strict form of this inequality coincides with the material stability condition of Chapter 9.

11.7 Equilibria with discontinuous deformation gradients

Phase transformations in so-called *shape-memory alloys* are characterized by deformations with discontinuous gradients. The characterization of such deformations and the conditions under which they arise constitute a major branch of research in nonlinear elasticity. The paper by Ball and James (1987) is essential reading in this area.

We know from our earlier discussion that if a deformation $\chi(\mathbf{x})$ is continuous with a gradient $\mathbf{F}(\mathbf{x})$ that jumps across one or more surfaces in the body, then discontinuities in the deformation gradient have the structure $[\mathbf{F}] = \mathbf{a} \otimes \mathbf{N}$ (cf. eqn (9.44)), where $\mathbf{N}$ is a unit normal to such a surface. If the deformation is equilibrated, then it is necessary that $\int_{\partial S} \mathbf{p} da = \mathbf{0}$, where S is any region in the configuration κ. We omit body forces, but they may easily be included without affecting our conclusions. We apply this to a pill-box of thickness h containing a surface of discontinuity with normal $\mathbf{N}$ (see Figure 11.2).

Because of the linearity of $\mathbf{p}$ with respect to the normal, the limit of $\mathbf{p}$ from the side of the surface with normal $-\mathbf{N}$ is $\mathbf{p}_-(-\mathbf{N}) = -\mathbf{p}_-(\mathbf{N})$. Letting h tend to zero, we then find that

$$\int_\pi [\mathbf{p}] da = \mathbf{0}, \tag{11.99}$$

where

$$[\mathbf{p}] = \mathbf{p}_+(\mathbf{N}) - \mathbf{p}_-(\mathbf{N}) \tag{11.100}$$

is the jump in traction and where π is an arbitrary subsurface of the discontinuity surface. The localization theorem yields the conclusion that

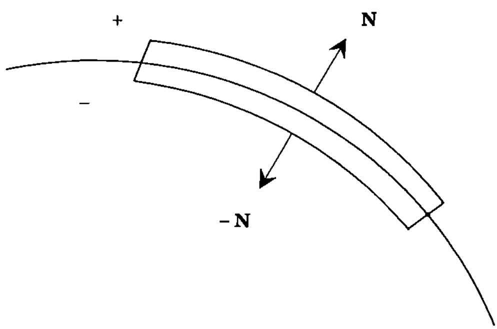

Figure 11.2 Pill-box surrounding a surface of discontinuity

$$[\mathbf{p}] = \mathbf{0} \tag{11.101}$$

at all points of the discontinuity surface. In the case of an unconstrained elastic material this is

$$\hat{\mathbf{P}}(\mathbf{F}_+)\mathbf{N} = \hat{\mathbf{P}}(\mathbf{F}_-)\mathbf{N}. \tag{11.102}$$

We seek conditions on the strain–energy function ensuring that this equality can be satisfied with $[\mathbf{F}] \neq \mathbf{0}$.

Fix a point on the surface of discontinuity and define

$$\mathbf{F}(u) = u\mathbf{F}_+ + (1-u)\mathbf{F}_- = \mathbf{F}_- + u\mathbf{a} \otimes \mathbf{N} \tag{11.103}$$

with $u \in [0,1]$. Let

$$\begin{aligned} f(u) &= \det \mathbf{F}(u) = \det \mathbf{F}_- \det(\mathbf{I} + u\mathbf{F}_-^{-1}\mathbf{a} \otimes \mathbf{N}) \\ &= \det \mathbf{F}_-(1 + u\mathbf{F}_-^{-1}\mathbf{a} \cdot \mathbf{N}). \end{aligned} \tag{11.104}$$

Observe that $f(0) = \det \mathbf{F}_- > 0$ and $f(1) = \det \mathbf{F}_+ > 0$. Then, because the graph of $f(u)$ is a straight line between these endpoints, we have $f(u) > 0$; therefore, $\mathbf{F}(u)$ belongs to the domain of W.

Next, define

$$g(u) = \mathbf{a} \otimes \mathbf{N} \cdot \hat{\mathbf{P}}(\mathbf{F}_- + u\mathbf{a} \otimes \mathbf{N}) = \mathbf{a} \cdot \hat{\mathbf{P}}(\mathbf{F}_- + u\mathbf{a} \otimes \mathbf{N})\mathbf{N}. \tag{11.105}$$

Then,

$$g(0) = \mathbf{a} \cdot \hat{\mathbf{P}}(\mathbf{F}_-)\mathbf{N} = \mathbf{a} \cdot \hat{\mathbf{P}}(\mathbf{F}_+)\mathbf{N} = g(1). \tag{11.106}$$

Because $g(u)$ is differentiable, by Rolle's theorem there is $u_0 \in [0,1]$ such that $g'(u_0) = 0$, i.e.,

$$0 = \mathcal{M}(\mathbf{F}(u_0))[\mathbf{a} \otimes \mathbf{N}] \cdot \mathbf{a} \otimes \mathbf{N} = \mathbf{a} \cdot \mathbf{A}(\mathbf{F}(u_0), \mathbf{N})\mathbf{a}, \tag{11.107}$$

and so there is a deformation gradient in the domain of W where the strong ellipticity condition is violated. Thus, if such a discontinuity is to exist, the strain energy cannot be a strongly elliptic *function*, i.e., it cannot be strongly elliptic at all points in its domain. Because strong ellipticity is necessary for the strict form of the rank-one convexity condition, it also cannot be a strictly rank-one convex function.

11.8 The Maxwell–Eshelby relation

Any deformation gradient occurring in an energy-minimizing deformation field must be such as to satisfy the rank-one convexity condition pointwise. Deformation gradients at which this condition is violated are relegated to sets of zero volume measure, such as discontinuity surfaces, where they can make no contribution to the overall energy. In particular, the limiting values $\mathbf{F}_\pm$ on either side of a discontinuity surface must satisfy rank-one convexity if the deformation field is such as to furnish a global minimum of the energy. For example,

$$W(\mathbf{F}_- + \mathbf{a} \otimes \mathbf{N}) - W(\mathbf{F}_-) \geq \mathbf{a} \cdot \hat{\mathbf{P}}(\mathbf{F}_-)\mathbf{N} \tag{11.108}$$

for all $\mathbf{a} \otimes \mathbf{N}$. Choosing $\mathbf{a} \otimes \mathbf{N} = [\mathbf{F}]$, we infer that

$$W(\mathbf{F}_+) - W(\mathbf{F}_-) \geq \mathbf{a} \cdot \hat{\mathbf{P}}(\mathbf{F}_-)\mathbf{N}. \tag{11.109}$$

In the same way,

$$W(\mathbf{F}_+ - \mathbf{a} \otimes \mathbf{N}) - W(\mathbf{F}_+) \geq -\mathbf{a} \cdot \hat{\mathbf{P}}(\mathbf{F}_+)\mathbf{N}, \tag{11.110}$$

or

$$W(\mathbf{F}_+) - W(\mathbf{F}_-) \leq \mathbf{a} \cdot \hat{\mathbf{P}}(\mathbf{F}_+)\mathbf{N}. \tag{11.111}$$

Invoking eqn (11.102), we arrive at the *Maxwell–Eshelby relation*

$$W(\mathbf{F}_+) - W(\mathbf{F}_-) = \hat{\mathbf{P}}(\mathbf{F}_\pm) \cdot (\mathbf{F}_+ - \mathbf{F}_-). \tag{11.112}$$

11.8.1 Example: alternating simple shear

Consider a deformation with piecewise uniform gradient

$$\mathbf{F}_\pm = \mathbf{I} + \gamma_\pm \mathbf{i} \otimes \mathbf{j} \tag{11.113}$$

with $\gamma_- = -\gamma_+$. These are simple shears of alternating sign. Here, $\mathbf{j}$ is the normal to the plane of shear and $\mathbf{i}$ is the direction of shear. We can imagine a laminate consisting of such alternating shears, extending over a volume of the body (Figure 11.3).

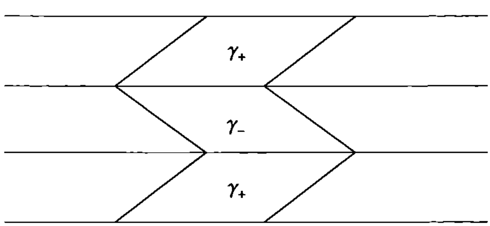

Figure 11.3 Alternating simple shear

We have

$$[\mathbf{F}] = [\gamma]\mathbf{i} \otimes \mathbf{j}, \tag{11.114}$$

and so $\mathbf{N} = \mathbf{j}$ and $\mathbf{a} = [\gamma]\mathbf{i}$, whereas the traction-continuity condition eqn (11.102) furnishes $[\mathbf{Pj}] = \mathbf{0}$ and, hence

$$[\tau] = 0, \tag{11.115}$$

where $\tau = \mathbf{i} \cdot \mathbf{Pj}$ is the shear stress on the plane of shearing.

The strain energy in simple shear is $w(\gamma) = W(\mathbf{I} + \gamma\mathbf{i} \otimes \mathbf{j})$. Thus,

$$\tau = \mathbf{P} \cdot \mathbf{i} \otimes \mathbf{j} = W_{\mathbf{F}} \cdot \mathbf{F}'(\gamma) = w'(\gamma), \tag{11.116}$$

and eqn (11.115) furnishes

$$\tau(\gamma_+) = \tau(\gamma_-). \tag{11.117}$$

The Maxwell–Eshelby relation reduces to

$$w(\gamma_+) - w(\gamma_-) = \mathbf{P}_\pm \cdot [\mathbf{F}] = [\gamma]\mathbf{i} \cdot \mathbf{P}_\pm \mathbf{j}. \tag{11.118}$$

This is Maxwell's *equal area rule*, in the form

$$\int_{\gamma_-}^{\gamma_+} \tau(\gamma)d\gamma = \tau^*(\gamma_+ - \gamma_-), \tag{11.119}$$

with $\tau^* = \tau(\gamma_\pm)$, requiring that the area under the shear stress vs amount-of-shear curve equal that of the rectangle with base $\gamma_+ - \gamma_-$ and height τ^*. If the material properties possess reflection symmetry with respect to the discontinuity surface, then $w(\gamma)$ is an even function and $\tau(\gamma)$ is odd.

The rank-one convexity condition implies that in each separate *phase* of uniform shear,

$$W[\mathbf{I} + (\gamma + \Delta\gamma)\mathbf{i} \otimes \mathbf{j}] - W(\mathbf{I} + \gamma\mathbf{i} \otimes \mathbf{j}) \geq \Delta\gamma\mathbf{i} \cdot \hat{\mathbf{P}}(\mathbf{I} + \gamma\mathbf{i} \otimes \mathbf{j})\mathbf{j}, \tag{11.120}$$

for all $\Delta\gamma$, or

$$w(\gamma + \Delta\gamma) - w(\gamma) \geq \Delta\gamma w'(\gamma), \tag{11.121}$$

so that any value of γ appearing in an energy-minimizing state belongs to a domain of convexity of the function $w(\cdot)$. The situation is then as depicted in Figure 11.4.

Shears $\gamma \in (\gamma_-, \gamma_+)$ are unstable and are thus excluded from the deformation field per se. Note that the response of the material in this case cannot distinguish between $w(\cdot)$ and its *convexification*, the lower convex envelope of $w(\cdot)$. A fuller discussion of this and similar problems may be found in Ericksen (1991).

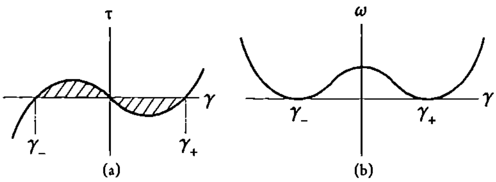

Figure 11.4 (a) Stress and (b) energy as functions of amount of shear.

Problem

We have seen that a compressible inviscid fluid is an elastic material with a strain-energy function of the form $W(\mathbf{F}) = w(J)$, with $J = \det \mathbf{F}$. Suppose the fluid is uniform in the sense that the same function pertains to every material point. The fluid fills a rigid container completely and no body forces are acting. Then, the potential energy of any configuration $\chi(\mathbf{x})$ of the fluid is

$$\mathcal{E}[\chi] = \int_{\kappa} W(\mathbf{F}(\mathbf{x}))dv, \tag{11.122}$$

where κ is the region enclosed by the rigid container.

(a) Prove that an equilibrium deformation $\chi(\mathbf{x})$ is a minimizer of the potential energy *if, and only if*, $J(\mathbf{x}) = det[\nabla\chi(\mathbf{x})]$ satisfies $w(\bar{J}) - w(J) \geq (\bar{J} - J)w'(J)$ for any $\bar{J} > 0$ and for all $\mathbf{x} \in \kappa$.

(b) Let $\nu = 1/\rho$ be the specific volume (volume per unit mass). Suppose that at a certain fixed temperature the constitutive function for the pressure, $p(\nu)$, is as depicted in Figure 11.5, wherein the shaded lobes have equal areas. Give a complete analysis of the stable equilibrium states of the fluid at the pressures p_a, p_b, and p_c. Identify those aspects of the state of the fluid that are uniquely determined in each case.

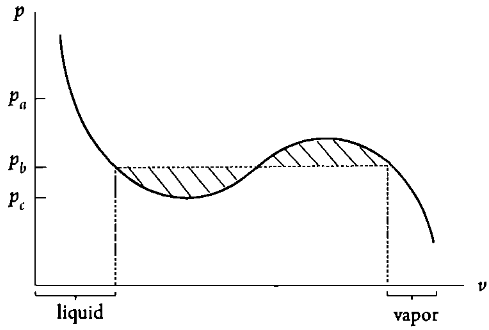

Figure 11.5 Non-monotone pressure-volume characteristic

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12

Linearized theory, the second variation and bifurcation of equilibria

We have already studied the linearized theory in some detail in Chapter 9. Here we elaborate on this theory in the case of equilibrium. That is, we study the linear theory of small equilibrium deformations superposed on a (finitely deformed) equilibrium state. This topic is often referred to as the theory of *small deformations superposed on large.* Ogden (1997) is the main source for this theory and should be consulted for further developments. Our purpose in discussing it is to outline a practical implementation of Euler's *non-uniqueness* criterion for potential instability, also known as *buckling*. This criterion is a cornerstone of engineering analysis.

In Chapter 9 we outlined the theory for incompressible materials. Here, we generalize by imposing a constraint of the form $\phi(\mathbf{F}) = 0$ on all admissible deformations. Let $\mathbf{y}_0(\mathbf{x})$ be an equilibrium deformation, and consider the static perturbation

$$\chi(\mathbf{x};\epsilon) = \mathbf{y}_0(\mathbf{x}) + \epsilon\mathbf{u}(\mathbf{x}) + \frac{1}{2}\epsilon^2\mathbf{v}(\mathbf{x}) + o(\epsilon^2) \tag{12.1}$$

with $\epsilon \in (-\epsilon_0, \epsilon_0)$ and $|\epsilon_0| \ll 1$. Here, $\mathbf{u} = \frac{\partial}{\partial\epsilon}\chi_{|\epsilon=0}$, $\mathbf{v} = \frac{\partial^2}{\partial\epsilon^2}\chi_{|\epsilon=0}$, etc. Then,

$$\mathbf{F}(\mathbf{x};\epsilon) = \mathbf{F}_0(\mathbf{x}) + \epsilon\nabla\mathbf{u}(\mathbf{x}) + \frac{1}{2}\epsilon^2\nabla\mathbf{v}(\mathbf{x}) + o(\epsilon^2), \tag{12.2}$$

and this must be such that $\phi(\mathbf{F}(\mathbf{x};\epsilon)) \equiv 0$ for all $\epsilon \in (-\epsilon_0, \epsilon_0)$. Accordingly,

$$0 = \phi'_{|\epsilon=0} = \phi_{\mathbf{F}}(\mathbf{F}_0)\cdot\nabla\mathbf{u}, \quad 0 = \phi''_{|\epsilon=0} = \phi_{\mathbf{FF}}(\mathbf{F}_0)[\nabla\mathbf{u}]\cdot\nabla\mathbf{u} + \phi_{\mathbf{F}}(\mathbf{F}_0)\cdot\nabla\mathbf{v}, \quad \text{etc.,} \tag{12.3}$$

which constitute constraints on $\nabla\mathbf{u}$ and $\nabla\mathbf{v}$. Note that there is no solution if $\chi(\mathbf{x};\epsilon)$ is linear in ϵ, i.e., if $\mathbf{v}(\mathbf{x})$ vanishes. For, frame-invariant constraints are inherently nonlinear (i.e; $\phi_{\mathbf{FF}} \neq \mathbf{0}$—see the Problem in Section 6.1) and a purely linear perturbation $\nabla\mathbf{u}$ is thus overdetermined by eqn (12.3).

The stress is

$$\mathbf{P} = \hat{\mathbf{P}}(\mathbf{F}) + \lambda\phi_{\mathbf{F}}(\mathbf{F}), \tag{12.4}$$

where $\hat{\mathbf{P}} = W_{\mathbf{F}}$ is the constitutive contribution and λ is a Lagrange multiplier. From eqn (12.2),

$$\hat{\mathbf{P}}(\mathbf{F}) = \hat{\mathbf{P}}(\mathbf{F}_0) + \epsilon\mathcal{M}(\mathbf{F}_0)[\nabla\mathbf{u}] + o(\epsilon) \tag{12.5}$$

and

$$\phi_{\mathbf{F}}(\mathbf{F}) = \phi_{\mathbf{F}}(\mathbf{F}_0) + \epsilon\phi_{\mathbf{FF}}(\mathbf{F}_0)[\nabla\mathbf{u}] + o(\epsilon). \tag{12.6}$$

We assume that

$$\lambda = \lambda_0 + \epsilon\lambda' + \frac{1}{2}\epsilon^2\lambda'' + o(\epsilon^2), \tag{12.7}$$

and conclude that

$$\mathbf{P} = \mathbf{P}_0 + \epsilon\mathbf{P}' + o(\epsilon), \tag{12.8}$$

where

$$\mathbf{P}_0 = \hat{\mathbf{P}}(\mathbf{F}_0) + \lambda_0\phi_{\mathbf{F}}(\mathbf{F}_0), \tag{12.9}$$

and

$$\mathbf{P}' = \mathcal{H}(\mathbf{F}_0; \lambda_0)[\nabla\mathbf{u}] + \lambda'\phi_{\mathbf{F}}(\mathbf{F}_0), \tag{12.10}$$

with

$$\mathcal{H}(\mathbf{F}_0; \lambda_0) = \mathcal{M}(\mathbf{F}_0) + \lambda_0\phi_{\mathbf{FF}}(\mathbf{F}_0) = (W + \lambda_0\phi)_{\mathbf{FF}|\mathbf{F}_0}. \tag{12.11}$$

Equilibrium without body force requires, of course, that $Div\mathbf{P} = \mathbf{0}$. Dividing by ϵ and passing to the limit, we derive

$$Div\mathbf{P}' = \mathbf{0}, \tag{12.12}$$

subject to

$$\phi_{\mathbf{F}}(\mathbf{F}_0) \cdot \nabla\mathbf{u} = 0. \tag{12.13}$$

This is a linear system for the fields $\mathbf{u}(\mathbf{x})$ and $\lambda'(\mathbf{x})$. In a mixed position/traction boundary-value problem, we assign $\mathbf{p}' = \mathbf{P}'\mathbf{N}$ on $\partial\kappa_p$ and $\mathbf{u}$ on its complement.

Problems

1. Suppose $\partial\kappa_p$ is loaded by a pressure of fixed intensity p. Show that

$$\mathbf{p}' = -p[(div\mathbf{u})\mathbf{I} - (grad\mathbf{u})^t]\mathbf{F}_0^*\mathbf{N}, \tag{12.14}$$

where *grad* and *div* are the gradient and divergence operators based on position $\mathbf{y}_0$.

2. Complete the differential equation $-gradq' = \dots$ for the incremental constraint pressure q' in an incompressible material.

Consider the potential energy of the deformation $\chi(\mathbf{x};\epsilon)$. We again confine attention to the mixed dead-load problem for the sake of illustration. The potential energy is the function of ϵ defined by

$$F(\epsilon) = \int_\kappa W(\mathbf{F}(\mathbf{x};\epsilon))dv - \int_{\partial\kappa_p} \mathbf{p}\cdot\chi(\mathbf{x};\epsilon)da, \tag{12.15}$$

where $\mathbf{p}$ is fixed, independent of ϵ. We assume that $\chi(\mathbf{x};\epsilon)$ is likewise fixed on the complement of $\partial\kappa_p$, and hence that $\mathbf{u}$ and $\mathbf{v}$ vanish there. Then,

$$F'(\epsilon) = \int_\kappa W_{\mathbf{F}}\cdot\mathbf{F}'(\epsilon)dv - \int_{\partial\kappa_p} \mathbf{P}_0\mathbf{N}\cdot\chi'(\epsilon)da. \tag{12.16}$$

Recalling that $\phi(\mathbf{F}(\epsilon)) \equiv 0$, we have $0 = \phi'(\epsilon) = \phi_{\mathbf{F}}\cdot\mathbf{F}'(\epsilon)$. Evaluating at $\epsilon = 0$ then yields

$$F'(0) = \int_\kappa [W_{\mathbf{F}}(\mathbf{F}_0) + g\phi_{\mathbf{F}}(\mathbf{F}_0)]\cdot\nabla\mathbf{u}dv - \int_{\partial\kappa} \mathbf{P}_0\mathbf{N}\cdot\mathbf{u}da, \tag{12.17}$$

for any scalar field $g(\mathbf{x})$. Identifying this with the equilibrium Lagrange-multiplier field $\lambda_0(\mathbf{x})$ and integrating by parts then furnishes

$$F'(0) = \int_\kappa \mathbf{P}_0\cdot\nabla\mathbf{u}dv - \int_\kappa Div(\mathbf{P}_0^t\mathbf{u})dv, \tag{12.18}$$

which reduces to

$$F'(0) = \int_\kappa \mathbf{u}\cdot Div\mathbf{P}_0 dv. \tag{12.19}$$

Because $\mathbf{P}_0$ is an equilibrium stress field, it nullifies the *first variation* $F'(0)$ and hence renders the potential energy stationary (cf. Problem no. 3 in Chapter 3).

Problem

Clearly, the first variation vanishes at an equilibrium state for all *variations* $\mathbf{u}$. In particular, the latter need not satisfy any equilibrium equations or boundary conditions beyond $\mathbf{u} = \mathbf{0}$ on the complement of $\partial\kappa_p$. Prove the converse, i.e, that if the first variation vanishes for *all* such variations, then the underlying state is in equilibrium.

Accordingly, the energy comparison reduces to

$$\mathcal{E}[\chi(\mathbf{x};\epsilon)] - \mathcal{E}[\mathbf{y}_0(\mathbf{x})] = \frac{1}{2}\epsilon^2[F''(0) + o(\epsilon^2)/\epsilon^2]. \tag{12.20}$$

Dividing by ϵ^2 and passing to the limit, we conclude that $\mathbf{y}_0$ is a stable deformation only if the *second variation* $F''(0)$ is non-negative; i.e.,

$$F''(0) \geq 0. \tag{12.21}$$

To make this explicit, we differentiate eqn (12.16), reaching

$$F''(\epsilon) = \int_\kappa \{W_{\mathbf{FF}}[\mathbf{F}'(\epsilon)] \cdot \mathbf{F}'(\epsilon) + W_{\mathbf{F}} \cdot \mathbf{F}''(\epsilon)\}dv - \int_{\partial\kappa} \mathbf{P}_0\mathbf{N} \cdot \chi''(\epsilon)da, \tag{12.22}$$

and

$$F''(0) = \int_\kappa \{\mathcal{M}(\mathbf{F}_0)[\nabla\mathbf{u}] \cdot \nabla\mathbf{u} + \hat{\mathbf{P}}(\mathbf{F}_0) \cdot \nabla\mathbf{v}\}dv - \int_{\partial\kappa} \mathbf{P}_0\mathbf{N} \cdot \mathbf{v}da. \tag{12.23}$$

Using

$$\int_\kappa \hat{\mathbf{P}}(\mathbf{F}_0) \cdot \nabla\mathbf{v}dv = \int_\kappa \{Div(\hat{\mathbf{P}}(\mathbf{F}_0)^t\mathbf{v}) - \mathbf{v} \cdot Div\hat{\mathbf{P}}(\mathbf{F}_0)\}dv, \tag{12.24}$$

together with $Div\hat{\mathbf{P}}(\mathbf{F}_0) = -Div\{\lambda_0\phi_{\mathbf{F}}(\mathbf{F}_0)\}$ (from $Div\mathbf{P}_0 = \mathbf{0}$), integrating the first term by parts, and invoking $\mathbf{v} = \mathbf{0}$ on the complement of $\partial\kappa_p$, we deduce that

$$F''(0) = \int_\kappa \{\mathcal{M}(\mathbf{F}_0)[\nabla\mathbf{u}] \cdot \nabla\mathbf{u} - \lambda_0\phi_{\mathbf{F}}(\mathbf{F}_0) \cdot \nabla\mathbf{v}\}dv. \tag{12.25}$$

Finally, we use eqns (12.3), part 2, and (12.11) to reduce this to

$$F''(0) = \int_\kappa \mathcal{H}(\mathbf{F}_0; \lambda_0))[\nabla\mathbf{u}] \cdot \nabla\mathbf{u}dv. \tag{12.26}$$

Using this expression it is possible to show that the Legendre-Hadamard inequality is a pointwise necessary condition for eqn (12.21). A simple proof may be found in the book by Fraejis de Veubeke (1979).

Problem

An elastic body is in frictionless contact with a rigid body on a part of its boundary. Give a direct proof (not relying on the 2nd variation) that a deformation minimizes the energy only if it exerts a pointwise compressive pressure distribution on the rigid body. *Hint*: To ensure that the elastic body and the rigid body do not inter-penetrate, variations $\mathbf{u}$ should be such that $\mathbf{u} \cdot \mathbf{n} \geq 0$ on the contacting surface, where $\mathbf{n}$ is the exterior unit normal field to the boundary of the rigid body.

Consider the linearized equilibrium problem with null incremental data, i.e.,

$$
\begin{aligned}
&Div\mathbf{P}' = \mathbf{0} \quad \text{and} \quad \phi_{\mathbf{F}}(\mathbf{F}_0)\cdot\nabla\mathbf{u} = 0 \quad \text{in} \quad \kappa, \quad \mathbf{P}'\mathbf{N} = \mathbf{0} \quad \text{on} \quad \partial\kappa_p, \\
&\mathbf{u} = \mathbf{0} \quad \text{on} \quad \partial\kappa\backslash\partial\kappa_p.
\end{aligned} \tag{12.27}
$$

Clearly, this admits $\mathbf{u} = \mathbf{0}$ and $\lambda' = 0$ as a solution no matter the values of the underlying deformation $\mathbf{y}_0$ and Lagrange multiplier λ_0. A *bifurcation* is a non-trivial solution $\{\mathbf{u}, \lambda'\}$ to the same problem. Its existence or otherwise depends on the underlying state. It corresponds to non-uniqueness of equilibrium in the linear approximation. For any bifurcation we have

$$
\begin{aligned}
0 &= \int_\kappa \mathbf{u}\cdot Div\mathbf{P}'dv = \int_\kappa (Div\{(\mathbf{P}')^t\mathbf{u}\} - \mathbf{P}'\cdot\nabla\mathbf{u})dv \\
&= \int_{\partial\kappa} \mathbf{P}'\mathbf{N}\cdot\mathbf{u}da - \int_\kappa \mathbf{P}'\cdot\nabla\mathbf{u}dv.
\end{aligned} \tag{12.28}
$$

Accordingly,

$$
\int_\kappa \mathcal{H}(\mathbf{F}_0;\lambda_0))[\nabla\mathbf{u}]\cdot\nabla\mathbf{u}dv = 0, \tag{12.29}
$$

and so bifurcations nullify the second variation of the energy. Taken together with eqn (12.21), this motivates the *Euler–Hill–Trefftz criterion*. Given $\{\mathbf{y}_0, \lambda_0\}$, if there is a non-zero $\{\mathbf{u}, \lambda'\}$ that furnishes a minimum value, namely zero, to the second variation of the energy, then the underlying state $\{\mathbf{y}_0, \lambda_0\}$ is potentially unstable. This may be cast as a variational problem subject only to eqn (12.27), part 2, and the requirement that $\mathbf{u} = \mathbf{0}$ on $\partial\kappa\backslash\partial\kappa_p$. Equations (12.27), parts 1 and 3, emerge as the Euler equation and natural boundary condition in this approach. This problem is of course linear, and thus far more tractable than the actual (nonlinear) problem. Ogden (1997) discusses a number of explicit applications of this criterion.

Note that the case of several simultaneous constraints is covered, rather obviously, by using eqn (6.10) in place of eqn (12.4) and repeating the argument leading to eqn (12.26), for all constraints acting simultaneously. The unconstrained case is recovered by suppressing eqn (12.13) and ignoring the Lagrange multipliers.

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FURTHER READING

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13

Elements of plasticity theory

Plasticity theory furnishes a foremost example of the utility of the concept of Elasticity in the formulation of more general models of material behavior. Roughly, plasticity theory seeks to describe the response of materials in which a strain persists after removal of load. This can occur when the load has reached a certain threshold. Various materials, such as metals, snow, plasticine, polymers, and paint come to mind. Existing theory pertains mainly to metals, for which the underlying mechanisms are reasonably well understood. If the metal is crystalline, with rows of lattice cells stacked one upon the other, and if a shear stress is applied in the axes of the lattice, then one typically observes a shear strain on these axes developing in response to the stress. If the shear stress meets or exceeds a critical value, then relative slipping of the stack ensues, producing a permanent macroscopic shear deformation. This is essentially a frictional effect, and hence invariably dissipative in nature, in contrast to pure elasticity. To describe it a suitable notion of energy dissipation will prove necessary.

The picture is similar in the case of simple tension of a bar (Figure 13.1). If we plot the mean cross-sectional axial stress (the axial force divided by the current cross-sectional area) against the current length of the bar, we typically see a response like that depicted in the figure. Upon initial loading, the length of the bar increases roughly in proportion to the stress. Their ratio is denoted by E. Further load or extension results in the onset of a nonlinear response, with the slope changing sharply and dropping significantly below E. If the load is reduced after the onset of this nonlinear regime, then the resulting deformation is quite different from that achieved by initial loading to the same stress level; the unloading typically is again linear, but somewhat offset relative to the initial loading curve. Reasoning as in Chapter 1, we are justified in attributing these observations to the material per se if the deformation and stress fields are uniform. In this case, the constant E is a material property, the famous *Young's modulus*. The mean stress is then equal to the local stress, and its value at the upper end of the linear part of the loading regime is the axial *yield stress*. All the while, the bar may shear or twist, while being extended or compressed, but here we focus attention on the relationship between uniaxial stress and length, l. As in the case of pure elasticity, the latter is normalized by initial length, l_0, yielding the usual *stretch* $\lambda(= l/l_0)$.

Finite Elasticity Theory. David J. Steigmann.

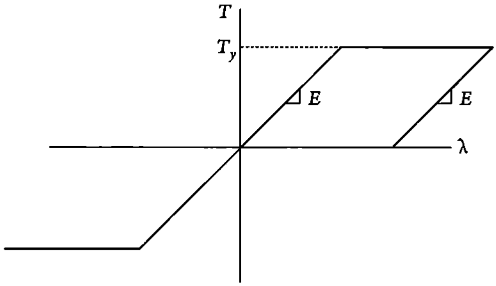

Figure 13.1 Idealized uniaxial response of an elastic-plastic material

13.1 Elastic and plastic deformations

If the load is removed after the onset of yield, then the length of the bar will recover, linearly, to a value l_i, as shown in the figure, in which the subscript is used to identify an *intermediate* configuration. This generates a so-called permanent stretch, or *plastic* stretch, given by $\lambda_p = l_i/l_0$, corresponding to a vanishing axial stress. The stretch λ just prior to the unloading is then given by

$$\lambda = \lambda_e \lambda_p, \tag{13.1}$$

where $\lambda_e = l/l_i$. This is called the *elastic* stretch, because it is that part of the total stretch required to restore the bar to length l under the application of the stress existing prior to the unloading. Thus, $\lambda_e = 1$ and $\lambda = \lambda_p$ when the stress vanishes. One thing worth noting is that the elasticity of the material—here characterized by the modulus E—is insensitive to the plastic deformation. Indeed, this conclusion extends to various other aspects of elastic response, as observed in famous experiments conducted by G.I. Taylor. We shall elaborate in due course. Attention is confined to the rate-independent theory, in which the response, as depicted in Figure 13.1, is either insensitive to the rate of deformation or the deformation proceeds so slowly that any rate dependence is not relevant.

One slightly unsatisfying aspect of this picture is that it mixes notions of stress and deformation together in describing the different types of stretch. In modern continuum mechanics we are accustomed to separating these notions for as long as possible so as to better understand the distinctions between kinematics and kinetics, deferring their intermingling to a separate class of *constitutive relations*, of which *elasticity* is, of course, a primary example. This issue has in fact been the source of much confusion over the course of the historical development of the subject of plasticity theory. Nevertheless, the different notions of stretch embodied in eqn (13.1) furnish a useful description of the underlying phenomena and, therefore, remain central to the subject. Our purpose in this chapter is to extend these ideas to general deformations and states of stress. We aim for a formulation of this important subject that is as clear and unambiguous as the modern theory of finite elasticity. Indeed, the motivation for this chapter stems from the conviction that such a development remains elusive to the present day.

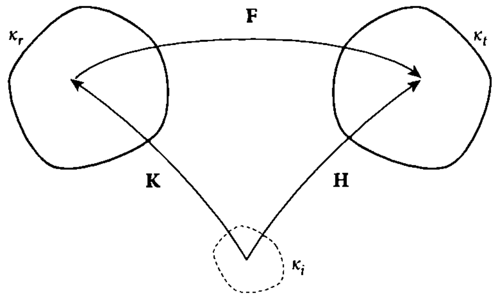

Figure 13.2 Elastic and plastic deformations

To begin, we introduce an *intermediate* configuration, κ_i, in which the material is presumed to be free of stress (see Figure 13.2). Let κ_r be a reference configuration, selected for convenience as per usual practice, and let κ_t be the configuration at the present time, t. The deformation from the reference configuration to the current has gradient $\mathbf{F}$ at time t and material point $\mathbf{x}$, as usual. Let $\mathbf{H}$ stand for the corresponding variable based on the use of κ_i as reference, and let $\mathbf{K}$ be that obtained when κ_i is used as reference and κ_t is replaced by κ_r. Then,

$$\mathbf{H} = \mathbf{FK}, \tag{13.2}$$

in which $\mathbf{F} = \nabla\chi$ and $\chi(\mathbf{x}, t)$ is the usual deformation. Because $\mathbf{H}$ is the value of $\mathbf{F}$ in the absence of plastic deformation ($\mathbf{K} = \mathbf{I}$), we assume that $J_H > 0$ and hence conclude that $J_K > 0$. Throughout this chapter we use the notation J_A to denote the determinant of a generic tensor $\mathbf{A}$. Comparing with eqn (13.1), we see that $\mathbf{H}$ corresponds to λ_e and $\mathbf{K}$ to λ_p^{-1}. In much of the literature eqn (13.2) is written as $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$, in which $\mathbf{F}_e(= \mathbf{H})$ and $\mathbf{F}_p(= \mathbf{K}^{-1})$ respectively are the *elastic* and *plastic* parts of the deformation. Here we use

$$\mathbf{G} = \mathbf{K}^{-1} \tag{13.3}$$

to denote the latter. It is important to note that while the factors in eqn (13.1) may be interchanged without loss of generality, this is not the case in eqn (13.2) for the simple reason that tensor multiplication does not commute.

In the course of extrapolating eqns (13.1) to (13.2) we have, of course assumed that κ_i is stress free. In the one-dimensional situation, the associated length l_i is achieved simply by removing the load. This corresponds to the removal of the stress pointwise in the case of uniform stress. However, pointwise removal of the stress is generally not feasible in the three-dimensional context. That is, it is *not* generally possible to have $\mathbf{T}(\mathbf{x}, t) = \mathbf{0}$ for all $\mathbf{x}$ in κ_r. In reality, there is a distribution of *residual stress* due to the presence of various defects in the body. These induce local lattice distortions in the case of crystalline metals, for example, which in turn manifest themselves as elastic strain and a consequent distribution of stress. This is typically the case even when the body is entirely unloaded, i.e., when no body forces are applied and the boundary tractions vanish.

Nevertheless, it is possible, in principle, to remove the *mean* stress via an *equilibrium* unloading process. In particular, in equilibrium the mean stress, $\bar{\mathbf{T}}$, is given by (see Chadwick, 1976)

$$vol(\kappa_t)\bar{\mathbf{T}} = \frac{1}{2}\int_{\partial\kappa_t}(\mathbf{t}\otimes\mathbf{y}+\mathbf{y}\otimes\mathbf{t})da + \frac{1}{2}\int_{\kappa_t}\rho(\mathbf{b}\otimes\mathbf{y}+\mathbf{y}\otimes\mathbf{b})dv, \tag{13.4}$$

where $\mathbf{t}$ and $\mathbf{b}$, respectively, are the boundary traction and body force. Accordingly, $\bar{\mathbf{T}} = \mathbf{0}$ if the entire body is unloaded.

In view of the Mean-Value Theorem for continuous functions (see Fleming, 1977), there exists $\bar{\mathbf{y}} \in \kappa_t$ such that $\mathbf{T}(\bar{\mathbf{y}}, t) = \bar{\mathbf{T}}$. Therefore, $\mathbf{T}(\bar{\mathbf{y}}, t) = \mathbf{0}$ for some $\bar{\mathbf{y}} \in \kappa_t$ if the body is unloaded and in equilibrium. Let

$$d(\kappa_t) = \sup_{\mathbf{y},\mathbf{z}\in\kappa_t}|\mathbf{y}-\mathbf{z}|. \tag{13.5}$$

This is the *diameter* of κ_t. Then, for every $\mathbf{y} \in \kappa_t$ we have $\mathbf{T}(\mathbf{y}, t) \to \mathbf{T}(\bar{\mathbf{y}}, t)$ as $d(\kappa_t) \to 0$. Accordingly, the local value of the stress can be made arbitrarily small as the diameter of the body is made to shrink to zero.

Of course, it is not possible to reduce the diameter of a given body to zero. However, we may regard any body as the union of an arbitrary number of arbitrarily small disjoint sub-bodies $P_t^{(n)}$, i.e., $\kappa_t = \cup_{n=1}^{\infty} P_t^{(n)}$, with $d(P_t^{(n)}) \to 0$. Imagine separating these sub-bodies and unloading them individually. We then have $\mathbf{T}(\mathbf{y}, t) \to \mathbf{0}$ for every $\mathbf{y} \in P_t^{(n)}$, for every n. Because every $\mathbf{y}$ in κ_t belongs to some $P_t^{(n)}$, this process results in a state in which the material is pointwise unstressed. Of course, each piece $P_t^{(n)}$ has in general experienced some (elastic) distortion in this process, and so the unstressed sub-bodies cannot be made congruent to fit together into a connected region of 3-space. Thus, there is no *global* stress-free configuration of the body and, hence, no position field χ_i, say, such that $d\chi_i = \mathbf{G}d\mathbf{x}$ (or $\mathbf{H}^{-1}d\mathbf{y}$); that is, there is no neighborhood in the vanishingly small unloaded sub-bodies that can be used to define a gradient of a position field. Accordingly, unlike $\mathbf{F}$, neither $\mathbf{G}$ nor $\mathbf{H}$ has the property of being a gradient. It follows that for any closed curve $\Gamma \subset \kappa_r$, with image $\gamma = \chi(\Gamma, t)$ in κ_t under the deformation map, the vector

$$\mathbf{B} = \int_{\Gamma}\mathbf{G}d\mathbf{x} = \int_{\gamma}\mathbf{H}^{-1}d\mathbf{y} \tag{13.6}$$

does not vanish. This is called the *Burgers vector* associated with the specified curve, induced by the plastic deformation.

In view of the foregoing, we regard κ_i as being associated with a material point $\mathbf{x}$, rather than as a configuration *per se*. It has the properties of a vector space. In fact it may be regarded as the tangent space to a certain body manifold, but this manifold is not Euclidean as it does not support a position field. This interpretation is the basis of an elegant differential-geometric theory of plastically deformed bodies (see the paper by Noll, 1967, and the book by Epstein and Elżanowski, 2007), which, however, is not emphasized here as it is largely superfluous as far as the formulation of problems is concerned.

13.2 Constitutive response

We have mentioned that the elastic properties of the material are essentially independent of the plastic deformation. This idea is codified in the expression

$$\mathcal{U}(P_t) = \int_{P_t} \psi(\mathbf{H})dv, \tag{13.7}$$

where ψ is the strain energy per unit volume of κ_t. This is determined entirely by the *elastic* deformation, in accordance with our hypothesis. In particular, this function is entirely unaffected by the relative slipping between adjacent lattice planes in a crystalline metal due to plastic deformation. This reflects the observation (see Batchelor, 1958) that relative slipping achieved without stress does not alter the structure of the lattice cells and so leaves the elastic constitutive response unchanged. In reality, slip is accomplished in steps, much as the overall displacement of a carpet is achieved by bunching it up locally and displacing the resulting bulges from one end of the carpet to the other. These stepwise displacements are called *dislocations*. They invariably generate localized lattice distortions and hence stresses. These are relieved, in principle, but cutting the body into small pieces, yielding the disjoint local intermediate configurations that we have identified with κ_i; accordingly, the latter are not realized in practice, but rather serve as a conceptual aid.

Here and henceforth, we shall confine attention to uniform materials for which the function ψ is not explicitly dependent on the material point; the strain-energy density is then given by one and the same function at all points of the body. This is the notion underlying Noll's theory of *materially uniform simple bodies*, which has had the most profound influence on the development of modern plasticity.

It proves convenient to base the theory on the strain energy "per unit volume" of κ_i. This may be defined unambiguously despite the fact that there is no global intermediate configuration and hence no associated volume per se; we simply multiply ψ by the local volume ratio induced by the deformation from κ_i to κ_t. This ratio is of course just J_H, which is well defined. The desired function is

$$W(\mathbf{H}) = J_H \psi(\mathbf{H}). \tag{13.8}$$

The use of this function, rather than ψ, affords a simple extension of what we have already learned in the case of pure elasticity. This follows from the fact that in the absence of plastic deformation—a situation we intend to encompass in the theory to be developed—$\mathbf{H}$ reduces to $\mathbf{F}$ and the energy W is then just the conventional strain energy per unit reference volume. In this specialization we have seen that the symmetry of the Cauchy stress—a restriction we impose a priori—is equivalent to the invariance of the energy under superposed rotations, i.e., $W(\mathbf{F}) = W(\mathbf{QF})$ for all rotations $\mathbf{Q}$. Because W is independent of $\mathbf{K}$ by construction, it follows that

$$W(\mathbf{H}) = W(\mathbf{QH}) \tag{13.9}$$

for all rotations $\mathbf{Q}$. This carries the same implications as in the case of pure elasticity, namely,

$$W(\mathbf{H}) = \hat{W}(\mathbf{C}_H), \tag{13.10}$$

where $\mathbf{C}_H = \mathbf{H}^t\mathbf{H}$ is the right Cauchy-Green tensor derived from $\mathbf{H}$. Equivalently, we may use

$$W(\mathbf{H}) = \tilde{W}(\mathbf{E}_H), \tag{13.11}$$

where $\mathbf{E}_H = \frac{1}{2}(\mathbf{C}_H - \mathbf{I})$ is the elastic Lagrange strain and $\tilde{W}(\mathbf{E}_H) = \hat{W}(2\mathbf{E}_H + \mathbf{I})$. As before, the Piola stress based on κ_i as reference is $W_{\mathbf{H}}$, and this is related to the 2nd Piola–Kirchhoff stress $\mathbf{S}_i$—also based on κ_i—by $W_{\mathbf{H}} = \mathbf{H}\mathbf{S}_i$, with

$$\mathbf{S}_i = \tilde{W}_{\mathbf{E}_H}. \tag{13.12}$$

The usual Cauchy stress $\mathbf{T}$ is given by

$$W_{\mathbf{H}} = \mathbf{T}\mathbf{H}^*. \tag{13.13}$$

Normally metals can undergo only small elastic strains before yielding, at least if the rate of strain is sufficiently small. We simplify the model accordingly by supposing that $|\mathbf{E}_H|$ is always small enough that the use of the quadratic-order approximation

$$\tilde{W}(\mathbf{E}_H) = \tilde{W}(\mathbf{0}) + \mathbf{E}_H \cdot \tilde{W}_{\mathbf{E}_H}(\mathbf{0}) + \frac{1}{2}\mathbf{E}_H \cdot \mathcal{C}[\mathbf{E}_H] + o(|\mathbf{E}_H|^2) \tag{13.14}$$

is justified, where

$$\mathcal{C} = \tilde{W}_{\mathbf{E}_H\mathbf{E}_H}(\mathbf{0}) \tag{13.15}$$

is the 4th-order tensor of elastic moduli, evaluated at zero strain. This possesses the major and minor symmetries discussed in Chapter 11. Because κ_i is associated with vanishing stress by assumption, the coefficient $\tilde{W}_{\mathbf{E}_H}(\mathbf{0})$ of the linear part of the expansion eqn (13.14) vanishes. Accordingly, the leading-order strain energy is purely quadratic:

$$\tilde{W}(\mathbf{E}_H) \simeq \frac{1}{2}\mathbf{E}_H \cdot \mathcal{C}[\mathbf{E}_H]. \tag{13.16}$$

This, of course, is just the usual elastic energy for small strains, yielding

$$\mathbf{S}_i = \mathcal{C}[\mathbf{E}_H]. \tag{13.17}$$

Relying on an observation made in Chapter 11, we take $\mathcal{C}$ to be positive definite ($\mathbf{A} \cdot \mathcal{C}[\mathbf{A}] > 0$ for all $\mathbf{A}$ with non-zero symmetric part), and conclude that $\tilde{W}$ is a convex function, i.e.,

$$\tilde{W}(\mathbf{E}_2) - \tilde{W}(\mathbf{E}_1) > \tilde{W}_{\mathbf{E}}(\mathbf{E}_1) \cdot (\mathbf{E}_2 - \mathbf{E}_1) \tag{13.18}$$

for all $\mathbf{E}_2 \neq \mathbf{E}_1$, in which the subscript $_H$ has been suppressed for the sake of clarity.

Because κ_i is free from elastic distortion, in the case of a crystalline metal the lattice is perfect and undistorted in κ_i. This has the consequence that

$$W(\mathbf{H}) = W(\mathbf{HR}), \tag{13.19}$$

for all *rotations* characterizing the symmetry of the lattice and, hence, that

$$\tilde{W}(\mathbf{E}_H) = \tilde{W}(\mathbf{R}^t\mathbf{E}_H\mathbf{R}). \tag{13.20}$$

We have seen that the collection of all such rotations is a group, the *symmetry group* of the lattice. For crystalline solids this group is always discrete, whereas for isotropic or transversely isotropic solids it is connected. In particular, isotropic materials satisfy eqn (13.20) for *all* rotations.

In the purely quadratic case, this has the well-known consequence that $\tilde{W}(\mathbf{E}_H)$ is of the form

$$\tilde{W}(\mathbf{E}_H) = \frac{1}{2}\lambda(tr\mathbf{E}_H)^2 + \mu\mathbf{E}_H \cdot \mathbf{E}_H, \tag{13.21}$$

in which λ and μ are the classical Lamé moduli. These are subject to the restrictions $\mu > 0$ and $3\lambda + 2\mu > 0$, which are necessary and sufficient for the positive definiteness of $\mathcal{C}$ in the present context. Using eqn (13.12), this in turn generates the classical stress-strain relation

$$\mathbf{S}_i = \lambda tr(\mathbf{E}_H)\mathbf{I} + 2\mu\mathbf{E}_H \tag{13.22}$$

for isotropic materials.

13.3 Energy and dissipation

It is convenient to adopt a referential description of the strain energy. Proceeding from eqns (13.7) and (13.8) we have

$$\mathcal{U}(P_t) = \int_{P_t} \psi(\mathbf{H})dv = \int_{P_r} J_F\psi\, dv = \int_{P_r} J_F J_H^{-1} W dv. \tag{13.23}$$

Thus,

$$\mathcal{U}(P_t) = \int_{P_r} \Psi(\mathbf{F}, \mathbf{K})dv, \tag{13.24}$$

where, from eqn (13.2),

$$\Psi(\mathbf{F},\mathbf{K}) = J_K^{-1} W(\mathbf{FK}). \tag{13.25}$$

The total mechanical energy in $P_t \subset \kappa_t$ is then given by

$$\mathcal{E}(P_t) = \int_{P_r} \Phi dv, \qquad \text{where} \quad \Phi = \Psi + \frac{1}{2}\rho_\kappa \left|\dot{\mathbf{y}}\right|^2, \tag{13.26}$$

where ρ_κ is the mass density in κ_r.

Problem

For fixed K, prove that Ψ is strongly elliptic at **F** if and only if W is strongly elliptic at **H**.

The power of the forces acting on P_t is

$$\mathcal{P}(P_t) = \int_{\partial P_r} \mathbf{p} \cdot \dot{\mathbf{y}} da + \int_{P_r} \rho_\kappa \mathbf{b} \cdot \dot{\mathbf{y}} dv. \tag{13.27}$$

Using the equation of motion eqn (2.28) we derive

$$\rho_\kappa \mathbf{b} \cdot \dot{\mathbf{y}} = \left(\frac{1}{2}\rho_\kappa \left|\dot{\mathbf{y}}\right|^2\right)^{\cdot} - \left[Div(\mathbf{P}^t\dot{\mathbf{y}}) - \mathbf{P} \cdot \nabla\dot{\mathbf{y}}\right]. \tag{13.28}$$

Substituting into eqn (13.27), applying the divergence theorem and using $\mathbf{p} \cdot \dot{\mathbf{y}} = \mathbf{P}^t\dot{\mathbf{y}} \cdot \mathbf{N}$, where **N** is the exterior unit normal to ∂P_r, we arrive at the *Mechanical Energy Balance* (cf. eqn (3.1))

$$\mathcal{P}(P_t) = \frac{d}{dt}\mathcal{K}(P_t) + \mathcal{S}(P_t), \tag{13.29}$$

where

$$\mathcal{K}(P_t) = \frac{1}{2}\int_{P_r} \rho_\kappa \left|\dot{\mathbf{y}}\right|^2 dv \tag{13.30}$$

is the kinetic energy and

$$\mathcal{S}(P_t) = \int_{P_r} \mathbf{P} \cdot \dot{\mathbf{F}} dv \tag{13.31}$$

is the stress power.

Next, we define the *dissipation* $\mathcal{D}$ to be the difference between the power supplied and the rate of change of the total energy; thus,

$$\mathcal{D}(P_t) = \mathcal{P}(P_t) - \frac{d}{dt}\mathcal{E}(P_t). \tag{13.32}$$

Using eqn (13.26) in the form $\mathcal{E}^{\cdot} = \mathcal{K}^{\cdot} + \mathcal{U}^{\cdot}$ and combining with eqn (13.32), it follows immediately that

$$\mathcal{D}(P_t) = \int_{P_r} D dv, \tag{13.33}$$

where

$$D = \mathbf{P} \cdot \dot{\mathbf{F}} - \dot{\Psi}. \tag{13.34}$$

In the purely elastic context we see from eqn (3.9) that D vanishes identically. Here, we impose the requirement $\mathcal{D}(P_t) \geq 0$ for all $P_t \subset \kappa_t$ and conclude, from the localization theorem, that

$$D \geq 0 \tag{13.35}$$

pointwise. This assumption serves as a surrogate for the 2nd law of thermodynamics in the present, purely mechanical, setting.

Problem

Suppose the state $\{\chi_\infty(\mathbf{x}), \mathbf{K}_\infty(\mathbf{x})\}$ is asymptotically stable relative to the static state $\{\chi_0(\mathbf{x}), \mathbf{K}_0(\mathbf{x})\}$ in the sense that any dynamical trajectory $\{\chi(\mathbf{x}, t), \mathbf{K}(\mathbf{x}, t)\}$ initiating at the latter tends to the former, pointwise, as $t \to \infty$. Show, for conservative problems, that the potential energy of the asymptotically stable state is no larger than that of the initial state.

To obtain a useful expression for the dissipation we proceed from eqn (13.25), obtaining

$$\dot{\Psi} = J_K^{-1}[\dot{W} - (\dot{J}_K/J_K)W]. \tag{13.36}$$

Here, we use the identity $\dot{J}_K/J_K = \mathbf{K}^{-t} \cdot \dot{\mathbf{K}}$ together with

$$\dot{W} = W_{\mathbf{H}} \cdot \dot{\mathbf{H}} = W_{\mathbf{H}}\mathbf{K}^t \cdot \dot{\mathbf{F}} + \mathbf{F}^t W_{\mathbf{H}} \cdot \dot{\mathbf{K}}. \tag{13.37}$$

Recalling that $W_{\mathbf{H}} = \mathbf{T}(\mathbf{FK})^* = \mathbf{PK}^*$ and hence that $W_{\mathbf{H}}\mathbf{K}^t = J_K\mathbf{P}$ and $\mathbf{F}^t W_{\mathbf{H}} = J_K\mathbf{F}^t\mathbf{PK}^{-t}$, eqn (13.36) is reduced to

$$\dot{\Psi} = \mathbf{P} \cdot \dot{\mathbf{F}} - \mathbb{E} \cdot \dot{\mathbf{K}}\mathbf{K}^{-1}, \tag{13.38}$$

where

$$\mathbb{E} = \Psi\mathbf{I} - \mathbf{F}^t\mathbf{P} \tag{13.39}$$

is Eshelby's *Energy-Momentum Tensor*. Accordingly, the local dissipation may be written in the form

$$D = \mathbb{E} \cdot \dot{\mathbf{K}}\mathbf{K}^{-1}. \tag{13.40}$$

This result, due in the present context to Epstein and Maugin (1995), highlights the role of the Eshelby tensor as the driving force for dissipation. We use it here, in conjunction with eqn (13.35), to derive restrictions on constitutive equations for the plastic evolution $\dot{\mathbf{K}}$. We note in passing, relying on eqn (13.38) and the chain rule, that

$$\mathbf{P} = \Psi_{\mathbf{F}}(\mathbf{F}, \mathbf{K}) \quad \text{and} \quad \mathbb{E} = -\Psi_{\mathbf{K}}(\mathbf{F}, \mathbf{K})\mathbf{K}^t. \tag{13.41}$$

The expression eqn (13.40) for D makes clear the fact that the dissipation vanishes in the absence of plastic evolution, i.e., $D = 0$ if $\dot{\mathbf{K}} = 0$. On the basis of empirical observation, we introduce the hypothesis that plastic evolution is inherently dissipative; thus, we suppose that $D \neq 0$ *if, and only if*, $\dot{\mathbf{K}} \neq \mathbf{0}$. In view of our previous assumption eqn (13.35), this means that

$$\dot{\mathbf{K}} \neq \mathbf{0} \quad \text{if and only if} \quad D > 0. \tag{13.42}$$

It may be observed from the definition eqn (13.39) that the Eshelby tensor is purely referential in nature, mapping the translation space of κ_r to itself. For reasons that will be explained later, it proves convenient to introduce a version of the Eshelby tensor, $\mathbb{E}_i$, that maps κ_i to itself. This is defined by the relation

$$\mathbb{E} = J_K^{-1}\mathbf{K}^{-t}\mathbb{E}_i\mathbf{K}^t. \tag{13.43}$$

Problems

1. Use eqn (13.39) to show that if $\mathbb{E}'$ is the Eshelby tensor derived by taking the current configuration as reference; i.e., $\mathbb{E}' = \psi\mathbf{I} - \mathbf{T}$, then $\mathbb{E} = J_F\mathbf{F}^t\mathbb{E}'\mathbf{F}^{-t}$. Thus, $\mathbb{E}$ is the *pullback* of $\mathbb{E}'$ from κ_t to κ_r. Show that $\mathbb{E}$ is the pullback of $\mathbb{E}_i$ from κ_i to κ_r, and that $\mathbb{E}_i$ is the pullback of $\mathbb{E}'$ from κ_t to κ_i.
2. Prove that

$$\mathbb{E}_i = W\mathbf{I} - \mathbf{H}^t W_{\mathbf{H}}. \tag{13.44}$$

   This implies that $\mathbb{E}_i$ is determined entirely by $\mathbf{H}$ and, hence, purely elastic in origin. Show that $\mathbb{E}_i = \hat{W}\mathbf{I} - \mathbf{C}_H\mathbf{S}_i$ and, hence, that $\mathbb{E}_i$ is insensitive to superposed rigid-body motions.
3. Prove, in the case of small elastic strain, that

$$\mathbb{E}_i = -\mathbf{S}_i + o(|\mathbf{E}_H|), \tag{13.45}$$

where $\mathbf{S}_i$ is given by eqn (13.17) and, hence, that the Eshelby tensor based on the intermediate configuration is given, to leading order and apart from sign, by the 2nd Piola–Kirchhoff stress referred to the same configuration.

4. Prove that

$$J_K D = \mathbb{E}_i \cdot \mathbf{K}^{-1}\dot{\mathbf{K}} \tag{13.46}$$

and hence that the assumption of inherent dissipativity is equivalent to the statement:

$$\dot{\mathbf{K}} \neq \mathbf{0} \quad \text{if and only if} \quad \mathbb{E}_i \cdot \mathbf{K}^{-1}\dot{\mathbf{K}} > 0. \tag{13.47}$$

It is interesting to observe that if $\mathbf{E}_H = \mathbf{0}$, then $W = 0$, $\mathbf{S}_i = \mathbf{0}$ and, hence, $D = 0$; then, eqn (13.47) implies that there can be no plastic evolution. That is, without stress, there can be no change in the plastic deformation. This is in accord with common observation.

13.4 Invariance

We have observed that the symmetry of the Cauchy stress is equivalent to the statement $W(\mathbf{H}) = W(\bar{\mathbf{Q}}\mathbf{H})$ for all rotations $\bar{\mathbf{Q}}$. Because the argument leading to this conclusion is purely local, the rotation $\bar{\mathbf{Q}}$ can conceivably vary from one material point to another. This stands in contrast to the rotation $\mathbf{Q}(t)$ associated with a superposed rigid-body motion, which must be spatially uniform and, hence, the same at all material points; here, we distinguish these cases explicitly in the notation.

In a superposed rigid-body motion, the deformation $\chi(\mathbf{x}, t)$ is changed to

$$\chi^+(\mathbf{x}, t) = \mathbf{Q}(t)\chi(\mathbf{x}, t) + \mathbf{c}(t) \tag{13.48}$$

for some spatially uniform vector function $\mathbf{c}$. It follows immediately—as we have seen—that $\mathbf{F}(= \nabla\chi)$ goes into $\mathbf{F}^+(= \nabla\chi^+)$, with $\mathbf{F}^+ = \mathbf{Q}\mathbf{F}$. The argument cannot be adapted to $\mathbf{H}$, however, because it is not the gradient of any position field.

Nevertheless, it follows that $\mathbf{H}^+ = \bar{\mathbf{Q}}\mathbf{F}\mathbf{K}$, whereas $\mathbf{H}^+ = \mathbf{F}^+\mathbf{K}^+ = \mathbf{Q}\mathbf{F}\mathbf{K}^+$. Consequently,

$$\bar{\mathbf{Q}}\mathbf{F}\mathbf{K} = \mathbf{Q}\mathbf{F}\mathbf{K}^+. \tag{13.49}$$

We would like to use this to arrive at some conclusion about the relationship between $\mathbf{K}^+$ and $\mathbf{K}$, but this requires a further hypothesis. A natural one is that the dissipation is insensitive to superposed rotations. To explore the implications we define $\mathbf{Z} = \mathbf{K}^+\mathbf{K}^{-1}$ and note, from (13.49), that $J_Z = 1$. Suppose $\mathbf{Z}(t_0) = \mathbf{I}$, so that the superposed rigid motion commences at time t_0. Using eqn (13.46) we find that the dissipation transforms to

$$J_K D^+ = \mathbb{E}_i^+ \cdot (\mathbf{K}^+)^{-1}\dot{\mathbf{K}}^+ = \mathbb{E}_i \cdot (\mathbf{K}^{-1}\mathbf{Z}^{-1}\dot{\mathbf{Z}}\mathbf{K} + \mathbf{K}^{-1}\dot{\mathbf{K}}) = J_K D + \mathbb{E}_i \cdot \mathbf{K}^{-1}\mathbf{Z}^{-1}\dot{\mathbf{Z}}\mathbf{K}, \tag{13.50}$$

wherein we have invoked the invariance of the Eshelby tensor $\mathbb{E}_i$. Accordingly, if $D^+ = D$ as assumed, then

$$\mathbb{E}_i \cdot \mathbf{K}^{-1}\mathbf{Z}^{-1}\dot{\mathbf{Z}}\mathbf{K} = 0, \tag{13.51}$$

and this purports to hold for $\mathbf{K}$ with $J_K > 0$. It, therefore, holds for $\mathbf{K} = \mathbf{I}$, yielding $\mathbf{Z} = \mathbf{K}^+$. This amounts to selecting κ_i as the reference configuration for the superposed rigid motion, this entailing no loss of generality as the argument is purely local. This $\mathbf{Z}$ is a *bona fide* plastic flow, and therefore subject to our strong dissipation hypothesis (13.47). This requires that $\dot{\mathbf{Z}}$ vanish, and hence, given the initial condition, that $\mathbf{K}^+ = \mathbf{K}$; thus, $\mathbf{G}^+ = \mathbf{G}$. From (13.49) it then follows that $\bar{\mathbf{Q}} = \mathbf{Q}(t)$. Altogether, then,

$$\mathbf{F}^+ = \mathbf{Q}\mathbf{F}, \quad \mathbf{H}^+ = \mathbf{Q}\mathbf{H} \quad \text{and} \quad \mathbf{K}^+ = \mathbf{K}. \tag{13.52}$$

13.5 Yielding, the work inequality and plastic flow

The situation depicted in Figure 13.1 suggests that the onset of yield may be characterized by the statement $|T| = T_Y$, where T_Y is the yield stress in uniaxial tension, a material property that may evolve with continued plastic flow. The inequality $|T| < T_Y$ is associated with elastic response, whereas $|T| > T_Y$ is impossible. Because of the one-one relation between stress and elastic stretch existing under our hypotheses, we could equally well describe yield using a relation of the form $f(\lambda_e) = 0$.

In the three-dimensional setting, we assume that the onset of yield may be described using a relation of the form $G(\mathbf{E}_H) = 0$, where G is an appropriate yield function pertaining to the material at hand. Of course, we may derive this from the more basic assumption that the yield function is dependent on $\mathbf{H}$, and that yield is insensitive to superposed rigid motions. Thus yield occurs when the elastic distortion lies on a certain manifold in 6-dimensional space. Again, because of the one-to-one relation between $\mathbf{E}_H$ and $\mathbf{S}_i$ under our hypotheses, we could equally well characterize yield in terms of the statement $F(\mathbf{S}_i) = 0$, where $F(\mathbf{S}_i) = G(\mathcal{C}^{-1}[\mathbf{S}_i])$ is the yield function, expressed in terms of the stress. We suppose elastic response to be operative when the stress satisfies $F(\mathbf{S}_i) < 0$, in which case the stress is said to belong to the *elastic range*, and that no state of stress existing in the material can be such that $F(\mathbf{S}_i) > 0$. We further suppose that $F(\mathbf{0}) < 0$ and, hence, that the stress-free state belongs to the elastic range. In this way we partition 6-dimensional stress space into the regions defined by positive, negative, and null values of F, with the first of these being inaccessible in any physically possible situation. This appears to disallow behavior of the kind associated with the Bauschinger effect, in which yield can occur upon load reversal before the unloaded state is attained. However, empirical facts support the view that this effect is accompanied by the emergence of dislocations, causing nonuniform distributions of stress and elastic strain in the material, which cannot be directly correlated with the overall global response represented in the test data. From this point of view, the Bauschinger effect is thus an artifact of the test being performed, not directly connected with constitutive properties per se.

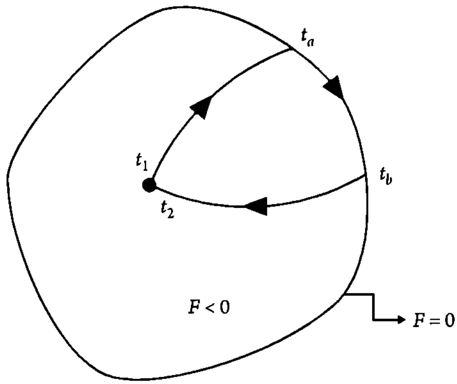

Figure 13.3 A cyclic process

Consider now a cyclic process, as described in Section 3.2 of Chapter 3. Reasoning as we did there, we have

$$\int_{t_1}^{t_2} \mathbf{P} \cdot \dot{\mathbf{F}} dt \geq 0, \tag{13.53}$$

where $t_{1,2}$, respectively, are the times when the cycle begins and ends. Suppose these times are such that the associated stresses satisfy $F < 0$; the cycle begins and ends in the elastic range (Figure 13.3).

Suppose the cycle is such that there exists a sub-interval of time $[t_a, t_b] \subset [t_1, t_2]$ during which $F = 0$, and that $F < 0$ outside this sub-interval. Then, we may have plastic flow, i.e., $\dot{\mathbf{K}} \neq \mathbf{0}$, during this sub-interval, while $\dot{\mathbf{K}} = \mathbf{0}$ outside it, implying that $\mathbf{K}(t_1) = \mathbf{K}(t_a)$ and $\mathbf{K}(t_2) = \mathbf{K}(t_b)$. Substituting eqn (13.34) and noting that the process is cyclic in the sense that $\mathbf{F}(t_2) = \mathbf{F}(t_1)$, we arrive at the statement

$$\Psi(\mathbf{F}(t_1), \mathbf{K}(t_b)) - \Psi(\mathbf{F}(t_1), \mathbf{K}(t_a)) + \int_{t_a}^{t_b} D dt \geq 0. \tag{13.54}$$

Equivalently,

$$\int_{t_a}^{t_b} [\Psi_{\mathbf{K}}(\mathbf{F}(t_1), \mathbf{K}(t)) \cdot \dot{\mathbf{K}}(t) + D(t)] dt \geq 0. \tag{13.55}$$

Dividing by $t_b - t_a (> 0)$ and passing to the limit, it follows from the mean value theorem that

$$\Psi_{\mathbf{K}}(\mathbf{F}(t_1), \mathbf{K}(t_a)) \cdot \dot{\mathbf{K}}(t_a) + D(t_a) \geq 0, \tag{13.56}$$

which may be written, using eqns (13.40) and (13.41), part 2, as

$$[\mathbb{E}(\mathbf{F}(t_a), \mathbf{K}(t_a)) - \mathbb{E}(\mathbf{F}(t_1), \mathbf{K}(t_a))] \cdot \dot{\mathbf{K}}(t_a)\mathbf{K}(t_a)^{-1} \geq 0. \tag{13.57}$$

Problem

Prove that this is equivalent to the inequality

$$[\mathbb{E}_i(\mathbf{E}_H(t_a)) - \mathbb{E}_i(\mathbf{E}_H(t_1))] \cdot \mathbf{K}(t_a)^{-1}\dot{\mathbf{K}}(t_a) \geq 0, \tag{13.58}$$

where $\mathbb{E}_i(\mathbf{E}_H)$ is the function of elastic strain obtained by recasting eqn (13.44). This means that the dissipation is maximized by states $\mathbf{E}_H$ (equivalently, $\mathbf{S}_i$) that lie on the yield surface.

In the case of small elastic strain, we substitute eqn (13.45) together with $\mathbf{K}^{-1}\dot{\mathbf{K}} = -\dot{\mathbf{G}}\mathbf{G}^{-1}$, which follows by differentiating $\mathbf{GK} = \mathbf{I}$, divide by $|\mathbf{E}_H|$, and pass to the limit in eqn (13.58) to derive the restriction

$$[\mathbf{S}_i(t_a) - \mathbf{S}_i(t_1)] \cdot Sym\dot{\mathbf{G}}(t_a)\mathbf{G}(t_a)^{-1} \geq 0, \tag{13.59}$$

in which we have inserted the qualifier *Sym* to reflect the fact that the term in the square brackets is symmetric; the inner product then picks up only the symmetric part of $\dot{\mathbf{G}}\mathbf{G}^{-1}$. We summarize this result in the statement:

$$(\mathbf{S} - \mathbf{S}^*) \cdot Sym\dot{\mathbf{G}}\mathbf{G}^{-1} \geq 0; \quad F(\mathbf{S}^*) \leq 0, \quad F(\mathbf{S}) = 0, \tag{13.60}$$

where the subscript $_i$ has been suppressed to promote clarity.

This inequality has a simple geometric interpretation having important implications for the structure of constitutive equations specifying the evolution of $\mathbf{G}$ (Figure 13.4). First, $Sym\dot{\mathbf{G}}\mathbf{G}^{-1}$ must be perpendicular to the tangent plane T_F to the yield surface at $\mathbf{S}$. Secondly, the entire elastic range, defined by $F < 0$, must lie to one side of T_F at $\mathbf{S}$. Thus, the elastic range is a convex set; that is, if $\mathbf{S}_{1,2}$ belong to the elastic range, then so does every point $u \in [0, 1]$ on the straight line $\mathbf{S}(u) = u\mathbf{S}_1 + (1 - u)\mathbf{S}_2$.

If $F(\mathbf{S})$ is a differentiable function, then T_F depends continuously on $\mathbf{S}$ and the surface $F(\mathbf{S}) = 0$ has exterior normal in the direction of the derivative $F_{\mathbf{S}}$ at point $\mathbf{S}$. We conclude that

$$Sym\dot{\mathbf{G}}\mathbf{G}^{-1} = \lambda F_{\mathbf{S}}, \tag{13.61}$$

for some scalar Lagrange multiplier field $\lambda(\mathbf{x}, t) \geq 0$, to be determined from the particular initial-boundary-value problem at hand. Precisely the same result is implied by the Kuhn–Tucker necessary conditions of optimization theory (see Zangwill, 1969). In the present setting, this pertains to the optimization problem:

$$\max(\mathbf{S} \cdot \dot{\mathbf{G}}\mathbf{G}^{-1}) \quad \text{subject to} \quad \mathbf{S} = \mathbf{S}^t \quad \text{and} \quad F(\mathbf{S}) \leq 0. \tag{13.62}$$

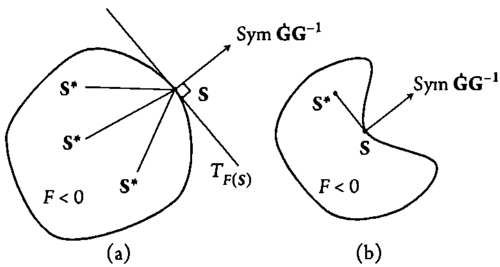

Figure 13.4 (a) Sym $\dot{G}G^{-1}$ is perpendicular to T_F at S. (b) The elastic range, $F < 0$, lies to one side of T_F at S

Plastic evolution is, therefore, such as to satisfy

$$\dot{\mathbf{G}}\mathbf{G}^{-1} = \lambda F_S + \mathbf{\Omega} \tag{13.63}$$

for some skew tensor $\mathbf{\Omega}(\mathbf{x}, t)$, called the *plastic spin.*

The foregoing considerations about yield and flow are quite general, and apply to both crystalline and non-crystalline materials. Modern theory for crystalline media is still in a state of active development (see Gurtin, Fried and Anand, 2010), particularly with respect to issues such as strain hardening—the evolution of the yield function with plastic flow—and plastic spin. In contrast, the classical theory, which purports to apply to isotropic materials, is well established and much simpler. However, although the associated literature is vast, it is seriously marred by the lack of any clear exposition of the explicit role played by (isotropic) material symmetry in the logical development of the subject. One of our main objectives here is to provide this missing link and, thus, to firmly establish the classical theory on the basis of the modern theory for finite elastic-plastic deformations. For all these reasons, attention is hereafter confined to the case of isotropy.

13.6 Isotropy

As we have seen, for isotropy the constitutive functions—exemplified above by the strain-energy function—must be insensitive to the replacement of $\mathbf{H}$ by $\mathbf{HQ}$, where $\mathbf{Q}$ is any rotation. We have seen that such invariance implies, in particular, that

$$\tilde{W}(\mathbf{E}_H) = \tilde{W}(\bar{\mathbf{E}}_H), \quad \text{where} \quad \bar{\mathbf{E}}_H = \mathbf{Q}^t\mathbf{E}_H\mathbf{Q} \tag{13.64}$$

is the rotated strain. In general, this yields

$$\bar{\mathbf{S}} = \mathbf{Q}^t\mathbf{S}\mathbf{Q}, \tag{13.65}$$

where $\bar{\mathbf{S}} = \tilde{W}_{\bar{\mathbf{E}}_H}$, which can easily be confirmed in the special case eqn (13.22) on replacing $\mathbf{E}_H$ by $\bar{\mathbf{E}}_H$.

Problem

Prove that this holds for isotropy in general.

With this result in hand, we are justified in requiring that the yield function, being a reflection of material properties, should satisfy the material symmetry restriction

$$F(\mathbf{S}) = F(\bar{\mathbf{S}}) \tag{13.66}$$

with $\bar{\mathbf{S}}$ given by eqn (13.65), for any rotation $\mathbf{Q}$. Accordingly, as in the preceding Problem,

$$F_{\bar{\mathbf{S}}} = \mathbf{Q}^t F_{\mathbf{S}} \mathbf{Q}. \tag{13.67}$$

Because of eqn (13.2), invariance statements of this kind are equivalent to the statement that scalar-valued constitutive functions should remain invariant if $\mathbf{K}$ is replaced by $\bar{\mathbf{K}} = \mathbf{KQ}$ - equivalently, if $\mathbf{G}$ is replaced by $\bar{\mathbf{G}} = \mathbf{Q}^t\mathbf{G}$, with $\mathbf{F}$ remaining fixed. To see how this replacement affects plastic flow, we compute

$$(\bar{\mathbf{G}})^{\cdot}\bar{\mathbf{G}}^{-1} = \mathbf{Q}^t\dot{\mathbf{G}}\mathbf{G}^{-1}\mathbf{Q} + \dot{\mathbf{Q}}^t\mathbf{Q}, \tag{13.68}$$

where we have allowed for the possibility that $\mathbf{Q}$ may be time-dependent. Substituting eqns (13.63) and (13.67) we conclude that

$$(\bar{\mathbf{G}})^{\cdot}\bar{\mathbf{G}}^{-1} = \lambda F_{\bar{\mathbf{S}}} + \mathbf{Q}^t(\boldsymbol{\Omega} + \mathbf{Q}\dot{\mathbf{Q}}^t)\mathbf{Q}. \tag{13.69}$$

Now, for any skew $\boldsymbol{\Omega}$ we can always find a rotation $\mathbf{Q}(t)$ to nullify the parenthetical term in eqn (13.69). To see this, suppose $\mathbf{B}(t)$ satisfies the initial-value problem

$$\dot{\mathbf{B}} = \mathbf{WB} \quad \text{with} \quad \mathbf{B}(0) = \mathbf{B}_0, \tag{13.70}$$

where $\mathbf{W}$ is skew and $\mathbf{B}_0$ is a rotation. Let $\mathbf{Z} = \mathbf{BB}^t$. Then,

$$\dot{\mathbf{Z}} = \mathbf{WZ} - \mathbf{ZW}, \quad \text{with} \quad \mathbf{Z}(0) = \mathbf{I}. \tag{13.71}$$

Clearly, a solution is furnished by $\mathbf{Z}(t) = \mathbf{I}$. A theorem on ordinary differential equations ensures that this is the only solution and, therefore, that $\mathbf{B}$ is necessarily orthogonal. Furthermore,

$$\dot{J}_B = \mathbf{B}^* \cdot \dot{\mathbf{B}} = J_B tr(\dot{\mathbf{B}}\mathbf{B}^{-1}), \tag{13.72}$$

and this vanishes because $\mathbf{W}$ is skew. Accordingly, $J_{B(t)} = J_{B(0)} = 1$, and $\mathbf{B}$ is a rotation. Because the rotation in eqn (13.69) is arbitrary, we are free to pick $\mathbf{Q} = \mathbf{B}$ (with $\mathbf{W} = \mathbf{\Omega}$, of course), to conclude that

$$(\bar{\mathbf{G}})^{\cdot}\,\bar{\mathbf{G}}^{-1} = \bar{\lambda} F_{\bar{\mathbf{S}}}, \quad \text{with} \quad \bar{\lambda} = \lambda. \tag{13.73}$$

Thus, by exploiting the degree of freedom afforded by the material symmetry group in the case of isotropy, we can effectively suppress plastic spin in the flow rule and use

$$\dot{\mathbf{G}}\mathbf{G}^{-1} = \lambda F_{\mathbf{S}}. \tag{13.74}$$

This is a major simplification that is not available in the case of crystalline materials.

Problem

Why not?

Before proceeding we pause to take note of an important empirical fact that applies with a high degree of accuracy to metals; namely, that yield is almost entirely insensitive to pressure. This is true in essentially all metals for pressures over a very large range that encompasses most applications. Thus, yield is insensitive to the value of $tr\mathbf{T}$, where $\mathbf{T}$ is the Cauchy stress.

Problem

Show that in the case of small elastic strain, $tr\mathbf{T} = tr\mathbf{S}_i + o(|\mathbf{E}_H|)$.

Thus, as the model we are pursuing purports to be valid to leading order in elastic strain, it follows that the yield function should be insensitive to $tr\mathbf{S}_i$. It should, therefore, depend on $\mathbf{S}_i$ entirely through its deviatoric part, $Dev\mathbf{S}_i$. Again, omitting the subscript, we write

$$F(\mathbf{S}) = \tilde{F}(Dev\mathbf{S}). \tag{13.75}$$

Problem

Show that $Dev\bar{\mathbf{S}} = \mathbf{Q}^t(Dev\mathbf{S})\mathbf{Q}$ and, hence, that

$$\tilde{F}(Dev\mathbf{S}) = \tilde{F}(\mathbf{Q}^t(Dev\mathbf{S})\mathbf{Q}). \tag{13.76}$$

Recall that in the theory for small elastic strains, we agreed to expand the strain–energy function up to quadratic order in the elastic strain. Moreover, the stress is approximated by an invertible, linear function of elastic strain. Accordingly, the strain energy may be regarded as a quadratic function of the stress $\mathbf{S}$. For consistency we also approximate the yield function by a quadratic function of the same stress. Because $Dev\mathbf{S}$ is a linear function of $\mathbf{S}$, this means that $\tilde{F}$ should be approximated by a quadratic function. The most general such function in the case of isotropy is a linear combination of $tr(Dev\mathbf{S})$, $(trDev\mathbf{S})^2$ and $tr(Dev\mathbf{S})^2 = |Dev\mathbf{S}|^2$, of which the first two vanish identically. The most general yield function of the required kind such that the yield surface $F = 0$ separates regions defined by $F < 0$ and $F > 0$ in stress space is then of the form

$$\tilde{F}(Dev\mathbf{S}) = \frac{1}{2}|Dev\mathbf{S}|^2 - k^2. \tag{13.77}$$

This is the famous yield function proposed by von Mises. The present derivation, based on material symmetry arguments in respect of an intermediate configuration, together with the assumption of differentiability of the yield function, promotes understanding of its position in the overall theory.

Because the set of symmetric tensors can be regarded as the direct sum of the 5-dimensional linear space of deviatoric tensors and the one-dimensional space of spherical tensors, it follows that the yield surface defined by $F = 0$ is a cylinder in 6-dimensional stress space of radius $\sqrt{2}k$. Here, k is the yield stress in shear. That is, if the state of stress is a pure shear of the form

$$\mathbf{S} = S(\mathbf{i} \otimes \mathbf{j} + \mathbf{j} \otimes \mathbf{i}), \tag{13.78}$$

with $\mathbf{i}$ and $\mathbf{j}$ orthonormal, then $|Dev\mathbf{S}|^2 = 2S^2$ and the onset of yield occurs when $|S| = k$. Here, k may be a fixed constant, corresponding to *perfect* plasticity, or may depend on appropriate variables that characterize the manner in which the state of the material evolves with plastic flow. The latter case refers to so-called *strain hardening*, the understanding of which is the central open problem of the phenomenological theory of plasticity.

The reader is likely aware that alternative yield functions, such as that associated with the name Tresca, are frequently used in the theory of plasticity for isotropic materials. This function, which we do not record here, is in fact nondifferentiable and, hence, inaccessible by the present line of reasoning. However, experiments conducted by Taylor and Quinney indicate that it is less accurate from the empirical point of view that the von Mises function, despite the seeming generality gained by relaxing the assumption of differentiability (see the paper by Taylor and Quinney in Taylor's Collected Works, 1958).

The theory is completed by substituting eqn (13.77) into eqn (13.74), to generate the flow rule for the plastic deformation. To this end we use eqns (13.75) and (13.77) with the chain rule, obtaining

$$\begin{aligned} F_S \cdot \dot{\mathbf{S}} = \dot{F} = (\tilde{F})^{\cdot} &= \frac{1}{2}(Dev\mathbf{S} \cdot Dev\mathbf{S})^{\cdot} \\ &= Dev\mathbf{S} \cdot (Dev\mathbf{S})^{\cdot} = Dev\mathbf{S} \cdot Dev\dot{\mathbf{S}} = Dev\mathbf{S} \cdot \dot{\mathbf{S}}, \end{aligned} \tag{13.79}$$

and, hence, $F_S = Dev\mathbf{S}$. Finally, eqn (13.74) provides von Mises' flow rule

$$\dot{\mathbf{G}}\mathbf{G}^{-1} = \lambda Dev\mathbf{S}_i. \tag{13.80}$$

This implies that J_G is fixed, and hence that no volume change is induced by plastic flow.

13.7 Rigid-plastic materials

The elastic strain is invariably small in metals under low strain-rate conditions because it is bounded by the diameter of the elastic range. If the overall strain is nevertheless large, then the main contribution to the strain comes from plastic deformation. In this case, it is appropriate to consider the idealization of zero elastic strain, which entails the restriction $\mathbf{H}^t\mathbf{H} = \mathbf{I}$. The elastic deformation is, therefore, a rotation field, which we denote by $\mathbf{R}$. Because the elastic strain vanishes identically, the strain energy is fixed in value and the stress is arbitrary, i.e.,

$$0 = \dot{W} = \mathbf{S}_i \cdot \dot{\mathbf{E}}_H, \quad \text{with} \quad \dot{\mathbf{E}}_H = \mathbf{0}. \tag{13.81}$$

Accordingly, at this level of the discussion $\mathbf{S}_i$ is an arbitrary symmetric tensor, constitutively unrelated to the deformation as in a rigid body, granted that it satisfies the yield criterion. Furthermore, $J_H = 1$ and the relation between the Cauchy stress and $\mathbf{S}_i$ reduces to

$$\mathbf{S}_i = \mathbf{R}^t\mathbf{T}\mathbf{R}, \tag{13.82}$$

and so

$$Dev\mathbf{S}_i = Dev(\mathbf{R}^t\mathbf{T}\mathbf{R}) = \mathbf{R}^t(Dev\mathbf{T})\mathbf{R}. \tag{13.83}$$

The yield function reduces to

$$\tilde{F}(Dev\mathbf{S}) = \tilde{F}(\mathbf{R}^t(Dev\mathbf{T})\mathbf{R}) = \tilde{F}(Dev\mathbf{T}), \tag{13.84}$$

the second equality being a consequence of isotropy, and is, therefore, expressible in terms of the Cauchy stress alone, as in the more conventional expositions of the classical theory.

Using eqn (13.83), we may cast the flow rule eqn (13.80) in the form

$$\mathbf{R}(\dot{\mathbf{G}}\mathbf{G}^{-1})\mathbf{R}^t = \lambda Dev\mathbf{T}. \tag{13.85}$$

We can express this in a more convenient and conventional form by using the well-known decomposition

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \tag{13.86}$$

of the spatial velocity gradient $\mathbf{L}$ into the sum of the straining tensor $\mathbf{D} = Sym\mathbf{L}$ and the vorticity tensor $\mathbf{W} = Skw\mathbf{L}$. Using $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ together with eqn (13.2), we find in the present specialization to $\mathbf{H} = \mathbf{R}$ that

$$\mathbf{L} = \dot{\mathbf{R}}\mathbf{R}^t + \mathbf{R}(\dot{\mathbf{G}}\mathbf{G}^{-1})\mathbf{R}^t \tag{13.87}$$

in which the first term is skew while the second, according to eqn (13.85), is symmetric. The uniqueness of the decomposition then yields $\mathbf{D} = \mathbf{R}(\dot{\mathbf{G}}\mathbf{G}^{-1})\mathbf{R}^t$ and, hence, the classical flow rule

$$\mathbf{D} = \lambda Dev\mathbf{T}, \tag{13.88}$$

due to St. Venant. The Cauchy stress is

$$\mathbf{T} = Dev\mathbf{T} - p\mathbf{I}, \tag{13.89}$$

where the pressure p is constitutively indeterminate. Equation (13.88) is the central equation of the classical theory and predates the modern theory for finite elastic–plastic deformations by at least a century. Its straightforward derivation via the modern theory, relying on simple ideas about material symmetry, brings unity and perspective to this most important branch of solid mechanics.

Problem

Show that the dissipation is $D = 2\lambda k^2$ and is, therefore, positive if and only if $\lambda > 0$.

13.8 Plane strain of rigid-perfectly plastic materials: slip-line theory

We consider deformations in the y_1, y_2—plane and, thus, confine attention to velocity fields of the form $\mathbf{v} = v_\alpha(y_1, y_2)\mathbf{e}_\alpha$. Then eqn (13.88) furnishes $Dev\mathbf{T} \equiv \tau = \tau_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$, implying that $p = -T_{33}$. The pressure field is equal to the confining stress required to maintain the plane-strain condition. Using $3p = -tr\mathbf{T}$ we conclude that $p = -\frac{1}{2}T_{\alpha\alpha}$.

13.8.1 State of stress, equilibrium

The yield criterion eqn (13.77), with eqn (13.84), reduces to

$$\begin{aligned} 2k^2 &= \tau_{\alpha\beta}\tau_{\alpha\beta} = (T_{\alpha\beta} + p\delta_{\alpha\beta})(T_{\alpha\beta} + p\delta_{\alpha\beta}) \\ &= T_{\alpha\beta}T_{\alpha\beta} + 2pT_{\alpha\alpha} + p^2\delta_{\alpha\alpha} \\ &= T_{\alpha\beta}T_{\alpha\beta} - 2p^2 = T_{\alpha\beta}T_{\alpha\beta} - \frac{1}{2}(T_{\alpha\alpha})^2, \end{aligned} \tag{13.90}$$

or

$$(T_{11} - T_{22})^2 + 4T_{12}^2 = 4k^2. \tag{13.91}$$

Problem

Show that the principal stresses are

$$T_1 = -p + k, \quad T_2 = -p - k \quad \text{and} \quad T_3 = -p. \tag{13.92}$$

We conclude that the stress state is

$$\mathbf{T} = -p\mathbf{I} + k(\mathbf{u}_1 \otimes \mathbf{u}_1 - \mathbf{u}_2 \otimes \mathbf{u}_2), \tag{13.93}$$

where $\{\mathbf{u}_i\}$, with $\mathbf{u}_3 = \mathbf{e}_3$, are the principal stress axes. Let $\mathbf{t}$ and $\mathbf{s}$ be orthonormal vector fields, such that

$$\mathbf{u}_1 = \frac{\sqrt{2}}{2}(\mathbf{s} + \mathbf{t}), \quad \mathbf{u}_2 = \frac{\sqrt{2}}{2}(\mathbf{s} - \mathbf{t}). \tag{13.94}$$

Then,

$$\mathbf{T} = -p\mathbf{I} + k(\mathbf{t} \otimes \mathbf{s} + \mathbf{s} \otimes \mathbf{t}), \tag{13.95}$$

which implies that k is the shear stress on the $\mathbf{s}, \mathbf{t}$ axes.

Problem

Show that $div(\mathbf{a} \otimes \mathbf{b}) = (grad\mathbf{a})\mathbf{b} + (div\mathbf{b})\mathbf{a}$, where *grad* and *div* are the gradient and divergence operators based on position $\mathbf{y}$.

Accordingly, in a perfectly plastic material ($k = const.$), equilibrium without body force requires that

$$grad(p/k) = (grad\mathbf{t})\mathbf{s} + (div\mathbf{s})\mathbf{t} + (grad\mathbf{s})\mathbf{t} + (div\mathbf{t})\mathbf{s}. \tag{13.96}$$

We define a field $\theta(\mathbf{x})$ such that

$$\mathbf{t} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, \quad \mathbf{s} = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2. \tag{13.97}$$

Then,

$$d\mathbf{t} = \mathbf{s}d\theta \quad \text{and} \quad d\mathbf{s} = -\mathbf{t}d\theta, \tag{13.98}$$

yielding

$$grad\mathbf{t} = \mathbf{s} \otimes grad\theta, \quad grad\mathbf{s} = -\mathbf{t} \otimes grad\theta \tag{13.99}$$

and, therefore,

$$div\mathbf{t} = \mathbf{s} \cdot grad\theta, \quad div\mathbf{s} = -\mathbf{t} \cdot grad\theta. \tag{13.100}$$

Substituting into eqn (13.96), we derive

$$grad(p/2k) = (\mathbf{s} \cdot grad\theta)\mathbf{s} - (\mathbf{t} \cdot grad\theta)\mathbf{t}, \tag{13.101}$$

which is equivalent to the two equations

$$\mathbf{t} \cdot grad(p/2k + \theta) = 0 \quad \text{and} \quad \mathbf{s} \cdot grad(p/2k - \theta) = 0, \tag{13.102}$$

due to Prandtl and Hencky. These require that $p/2k \pm \theta$ take constant values on the trajectories defined by $dy_2/dy_1 = \tan\theta, -\cot\theta$, respectively. The latter are the *characteristic curves* of the *hyperbolic* system of PDEs for the fields p and θ. Remarkably, the stress is *statically determinate*, i.e., granted suitable boundary conditions, it can be determined without knowledge of the deformation. These striking features of the theory of perfectly plastic materials contrast sharply with the mathematical setting of the theory of elasticity.

13.8.2 Velocity field

It proves advantageous to decompose the velocity field in the (variable) basis $\{\mathbf{s}, \mathbf{t}\}$. Thus,

$$\mathbf{v} = v_t\mathbf{t} + v_s\mathbf{s}. \tag{13.103}$$

To compute the velocity gradient, we combine the chain rule with eqn (13.98) to obtain

$$\begin{aligned} d\mathbf{v} &= dv_t\mathbf{t} + v_t\mathbf{s}d\theta + dv_s\mathbf{s} - v_s\mathbf{t}d\theta \\ &= (gradv_t \cdot d\mathbf{y})\mathbf{t} + v_t\mathbf{s}(grad\theta \cdot d\mathbf{y}) + (gradv_s \cdot d\mathbf{y})\mathbf{s} - v_s\mathbf{t}(grad\theta \cdot d\mathbf{y}) \\ &= \mathbf{L}d\mathbf{y}, \end{aligned} \tag{13.104}$$

and conclude that

$$\mathbf{L} = \mathbf{t} \otimes gradv_t + v_t\mathbf{s} \otimes grad\theta + \mathbf{s} \otimes gradv_s - v_s\mathbf{t} \otimes grad\theta. \tag{13.105}$$

Then,

$$\begin{aligned} 2\mathbf{D} = {} & \mathbf{t} \otimes gradv_t + gradv_t \otimes \mathbf{t} + \mathbf{s} \otimes gradv_s + gradv_s \otimes \mathbf{s} \\ & v_t(\mathbf{s} \otimes grad\theta + grad\theta \otimes \mathbf{s}) - v_s(\mathbf{t} \otimes grad\theta + grad\theta \otimes \mathbf{t}). \end{aligned} \tag{13.106}$$

Using eqns (13.88) and (13.95) leads to

$$\mathbf{t} \cdot \mathbf{D}\mathbf{t} = 0 \quad \text{and} \quad \mathbf{s} \cdot \mathbf{D}\mathbf{s} = 0, \tag{13.107}$$

which together imply that the deformation is isochoric and the extension rates vanish along the directions **t** and **s**.

Problem

Show that eqn (13.107) are equivalent to the pair

$$\mathbf{t} \cdot (grad v_t - v_s grad\theta) = 0 \quad \text{and} \quad \mathbf{s} \cdot (grad v_s + v_t grad\theta) = 0. \tag{13.108}$$

These are the celebrated Geiringer equations. They are linear PDEs for the components v_t and v_s if the stress field is known.

Suppose the normal velocity v_s (resp., v_t) is continuous across the trajectory with unit-tangent field **t** (respectively, **s**). This means that fissures do not form in the material. Taking jumps, the first (resp. second) equation implies that $\mathbf{t} \cdot grad[v_t]$ (respectively, $\mathbf{s} \cdot grad[v_s]$) vanishes on this trajectory, so that the *slip* $[v_t]$ (respectively, $[v_s]$), if non-zero, is uniform along it. Hence, the name *slip-line fields* given to this subject.

The literature on this topic is vast. The books by Hill (1950), Kachanov (1974), and Johnson, Sowerby and Haddow (1970) and the article by Geiringer (1973) describe further theory and many worked-out solutions. Numerical solutions are discussed in the article by Collins (1982).

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1. The cofactor

Consider a vector-valued map that takes $\mathbf{a} \times \mathbf{b}$ into $\mathbf{Aa} \times \mathbf{Ab}$. If we can construct a *linear* map of this kind, then there is a tensor $\mathbf{A}^*$ such that $\mathbf{A}^*(\mathbf{a} \times \mathbf{b}) = \mathbf{Aa} \times \mathbf{Ab}$. $\mathbf{A}^*$ is called the *cofactor* of $\mathbf{A}$. This would qualify as a working definition of the cofactor, provided it could be shown that such a linear map exists. An elegant proof is given in the appendix of Chadwick (1976).

Our approach will be to simply assume linearity, construct a representation $A^*_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ for $\mathbf{A}^*$, and use it to confirm linearity after the fact. We have

$$\mathbf{Aa} \times \mathbf{Ab} = e_{kij}A_{il}A_{jm}a_l b_m \mathbf{e}_k, \tag{1.1}$$

where e_{ijk} is the permutation symbol. Write $\mathbf{A}^*(\mathbf{a} \times \mathbf{b}) = \mathbf{A}^*(e_{jlm}a_l b_m \mathbf{e}_j) = A^*_{kj}e_{jlm}a_l b_m \mathbf{e}_k$. Using the fact that $\{\mathbf{e}_i\}$ is a basis, we find that the original equation is equivalent to

$$A^*_{kj}e_{jlm}a_l b_m = e_{kij}A_{il}A_{jm}a_l b_m. \tag{1.2}$$

However, the a_l and b_m are arbitrary real numbers, so this must be satisfied no matter how we choose them. Pick $b_m = \delta_{1m}, \delta_{2m}, \delta_{3m}$ in succession, where δ_{jp} is the Kronecker delta. We get $A^*_{kj}e_{jlr}a_l = e_{kij}A_{il}A_{jr}a_l$. Now pick $a_l = \delta_{1l}, \delta_{2l}, \delta_{3l}$ in succession. This yields $A^*_{kj}e_{jqr} = e_{kij}A_{iq}A_{jr}$. Note that the left-hand side is skew in the subscripts q, r. For our result to make sense, the right-hand side had better be also (check: $e_{kij}A_{ir}A_{jq} = e_{kji}A_{jr}A_{iq} = e_{kji}A_{iq}A_{jr} = -e_{kij}A_{iq}A_{jr}$). Now multiply through by e_{pqr} and sum on q, r. Use the fact that $e_{jqr}e_{pqr} = 2\delta_{jp}$ to get:

$$A^*_{kp} = \frac{1}{2}e_{kij}e_{pqr}A_{iq}A_{jr}. \tag{1.3}$$

This formula was derived by making convenient choices of the vector components a_l, b_m, i.e. we have shown that it is a *necessary* condition for the definition to be true. To show that it is also sufficient, we must substitute into the left-hand side of eqn (1.2) and show that we get the right-hand side, for *any* $a_l b_m$.

Problem

Do so.

We have constructed $\mathbf{A}^* = A^*_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ such that $\mathbf{A}^*(\mathbf{a} \times \mathbf{b}) = \mathbf{Aa} \times \mathbf{Ab}$ for all $\mathbf{a}, \mathbf{b}$ and $\mathbf{A}^*$ does not depend on $\mathbf{a}, \mathbf{b}$. The function $\mathbf{f}(\mathbf{v}) = \mathbf{A}^*\mathbf{v}$ is linear and vector valued, and so the cofactor is a tensor.

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2. Gradients of scalar-valued functions of tensors

The gradient of a scalar-valued function of tensors is defined in exactly the same way as for functions of vectors or positions. Let $g(\mathbf{A})$ be such a function and suppose it is differentiable at $\mathbf{A}_1$. This means that for each $\mathbf{A}_2$ in an open set in *Lin* containing $\mathbf{A}_1$, there is a *linear* function $f(\mathbf{B})$, depending parametrically on the fixed tensor $\mathbf{A}_1$, such that

$$g(\mathbf{A}_2) = g(\mathbf{A}_1) + f(\mathbf{A}_2 - \mathbf{A}_1) + o(|\mathbf{A}_2 - \mathbf{A}_1|). \tag{2.1}$$

Because $f(\mathbf{B})$ is linear, it is expressible as the inner product of a unique tensor with $\mathbf{B}$; we call this tensor $\nabla g(\mathbf{A}_1)$. Often the notation $g_{\mathbf{A}}(\mathbf{A}_1)$ is used to make the independent variable explicit. Thus,

$$g(\mathbf{A}_2) = g(\mathbf{A}_1) + \nabla g(\mathbf{A}_1) \cdot (\mathbf{A}_2 - \mathbf{A}_1) + o(|\mathbf{A}_2 - \mathbf{A}_1|). \tag{2.2}$$

Using a mixed basis for illustrative purposes, let $\mathbf{A} = A_{iB}\mathbf{e}_i \otimes \mathbf{E}_B$. Then $\mathbf{A}_{1,2} = A_{iB}^{(1,2)}\mathbf{e}_i \otimes \mathbf{E}_B$. Let $\bar{g}(A_{jC}) = g(A_{iB}\mathbf{e}_i \otimes \mathbf{E}_B)$; then, eqn (2.2) may be written

$$\bar{g}(A_{jC}^{(2)}) = \bar{g}(A_{jC}^{(1)}) + (A_{iB}^{(2)} - A_{iB}^{(1)})\mathbf{e}_i \otimes \mathbf{E}_B \cdot \nabla g(\mathbf{A}_1) + o(|\mathbf{A}_2 - \mathbf{A}_1|). \tag{2.3}$$

This must hold for all $\mathbf{A}_{1,2}$. Imposing it for $\mathbf{A}_2 - \mathbf{A}_1 = A\mathbf{e}_1 \otimes \mathbf{E}_2$, for example $(A_{iB}^{(2)} - A_{iB}^{(1)} = A\delta_{i1}\delta_{B2})$, yields

$$\bar{g}(A_{jC}^{(1)} + A\delta_{j1}\delta_{C2}) - \bar{g}(A_{jC}^{(1)}) = A\mathbf{e}_1 \otimes \mathbf{E}_2 \cdot \nabla g(\mathbf{A}_1) + o(A). \tag{2.4}$$

Dividing by A and passing to the limit, we get

$$\mathbf{e}_1 \cdot [\nabla g(\mathbf{A}_1)]\mathbf{E}_2 = \frac{\partial \bar{g}}{\partial A_{12}}\Big|_{\mathbf{A}_1}, \tag{2.5}$$

wherein we hold fixed all components other than A_{12}. In general we then have

$$\nabla g(\mathbf{A}) = \frac{\partial \bar{g}}{\partial A_{iB}}\mathbf{e}_i \otimes \mathbf{E}_B \tag{2.6}$$

provided that all the derivatives are *independent*. This would not be the case if there were any *a priori* relation among the components, as is the case for symmetric or skew tensors, for example.

3. Chain rule

Consider a curve in *Lin* described by a differentiable function $\mathbf{A}(t)$ where t is a parameter in some open interval (a, b). Let $\tilde{g}(t) = g(\mathbf{A}(t))$. Suppose that g is differentiable with respect to $\mathbf{A}$ and that $\mathbf{A}$ is differentiable with respect to t. Furthermore, let $\mathbf{A}_{1,2} = \mathbf{A}(t_{1,2})$. Then $\tilde{g}(t)$ is differentiable and, from eqn (2.2) above,

$$\tilde{g}(t_2) = \tilde{g}(t_1) + \nabla g(\mathbf{A}_1) \cdot (\mathbf{A}_2 - \mathbf{A}_1) + o(|\mathbf{A}_2 - \mathbf{A}_1|). \tag{3.1}$$

We also have

$$\mathbf{A}_2 - \mathbf{A}_1 = (t_2 - t_1)\dot{\mathbf{A}}(t_1) + o(t_2 - t_1), \tag{3.2}$$

and, therefore,

$$|\mathbf{A}_2 - \mathbf{A}_1| = O(t_2 - t_1). \tag{3.3}$$

Thus,

$$\tilde{g}(t_2) - \tilde{g}(t_1) = (t_2 - t_1)\nabla g(\mathbf{A}_1) \cdot \dot{\mathbf{A}}(t_1) + o(t_2 - t_1). \tag{3.4}$$

Dividing by $t_2 - t_1$ and passing to the limit, we obtain the chain rule:

$$\dot{g} = \nabla g(\mathbf{A}) \cdot \dot{\mathbf{A}}. \tag{3.5}$$

In the text we use the notation

$$dg = \nabla g(\mathbf{A}) \cdot d\mathbf{A}. \tag{3.6}$$

4. Gradients of the principal invariants of a symmetric tensor

We need formulas for the gradients of the invariants $I_k(\mathbf{A})$ with respect to $\mathbf{A}$. We assume the tensor $\mathbf{A}$ to be symmetric so that its off-diagonal components are not independent. This means that a formula like eqn (2.6) above is not applicable; we resort to an alternative method based on the chain rule.

Let $\mathbf{A}(t)$ describe a curve in *Sym*, and consider

$$I_1(\mathbf{A}) = tr\mathbf{A} = \mathbf{I} \cdot \mathbf{A}. \tag{4.1}$$

Then,

$$\nabla I_1(\mathbf{A}) \cdot \dot{\mathbf{A}} = \dot{I}_1 = \mathbf{I} \cdot \dot{\mathbf{A}}. \tag{4.2}$$

The symmetry of $\mathbf{A}(t)$ implies that $\dot{\mathbf{A}}$ is symmetric too (the proof is immediate). If we decompose the tensor $\nabla I_1(\mathbf{A})$ into the sum of symmetric and skew parts, and then form the inner product with $\dot{\mathbf{A}}$, we find that

$$\nabla I_1(\mathbf{A}) \cdot \dot{\mathbf{A}} = (Sym\nabla I_1(\mathbf{A})) \cdot \dot{\mathbf{A}}, \tag{4.3}$$

where

$$2Sym\mathbf{T} = \mathbf{T} + \mathbf{T}^t \tag{4.4}$$

for any tensor $\mathbf{T}$. Then eqn (4.2) yields

$$[(Sym\nabla I_1(\mathbf{A})) - \mathbf{I}] \cdot \dot{\mathbf{A}} = 0 \tag{4.5}$$

for all symmetric $\dot{\mathbf{A}}$. Now the term in brackets is a symmetric tensor, and the condition says that it is orthogonal to every element in the set of symmetric tensors. That this set is a linear space follows from the fact that an arbitrary linear combination of symmetric tensors is symmetric and the set also contains the zero tensor. Therefore, the term in brackets must be the zero tensor, yielding

$$Sym\nabla I_1(\mathbf{A}) = \mathbf{I}. \tag{4.6}$$

Note that the derivation yields no information about the skew part of $\nabla I_1(\mathbf{A})$, which may be arbitrary. It is very common to simply set the skew part to zero, and to equate $\nabla I_1(\mathbf{A})$ to $Sym\nabla I_1(\mathbf{A})$. In particular, it is impossible to determine the skew part from the analysis; however, it is quite unnecessary to do so. This convention extends to any scalar field defined on the linear space of symmetric tensors.

Next, consider

$$I_2(\mathbf{A}) = tr\mathbf{A}^*. \tag{4.7}$$

Problem

Prove that $tr\mathbf{A}^* = \frac{1}{2}\left[(tr\mathbf{A})^2 - tr(\mathbf{A}^2)\right]$.

Then,

$$I_2(\mathbf{A}) = \frac{1}{2}(I_1^2 - \mathbf{I} \cdot \mathbf{A}^2), \tag{4.8}$$

and

$$\nabla I_2(\mathbf{A}) \cdot \dot{\mathbf{A}} = \dot{I}_2 = I_1\dot{I}_1 - \frac{1}{2}\mathbf{I} \cdot (\dot{\mathbf{A}}\mathbf{A} + \mathbf{A}\dot{\mathbf{A}}). \tag{4.9}$$

Using the trace definition of the inner product we can show that

$$\mathbf{I} \cdot (\dot{\mathbf{A}}\mathbf{A}) = \mathbf{I} \cdot (\mathbf{A}\dot{\mathbf{A}}) = \mathbf{A} \cdot \dot{\mathbf{A}}. \tag{4.10}$$

Thus,

$$[Sym\nabla I_2(\mathbf{A}) - (I_1\mathbf{I} - \mathbf{A})] \cdot \dot{\mathbf{A}} = 0 \tag{4.11}$$

for all symmetric $\dot{\mathbf{A}}$, yielding

$$Sym\nabla I_2(\mathbf{A}) = I_1\mathbf{I} - \mathbf{A}. \tag{4.12}$$

Finally, recall the standard result

$$\dot{J} = \mathbf{F}^* \cdot \dot{\mathbf{F}}, \tag{4.13}$$

where $\mathbf{F}^*$ is the cofactor of $\mathbf{F}$ and $J = \det \mathbf{F}$. By the same reasoning, with

$$I_3(\mathbf{A}) = \det \mathbf{A} \tag{4.14}$$

we get

$$[Sym\nabla I_3(\mathbf{A}) - \mathbf{A}^*] \cdot \dot{\mathbf{A}} = 0, \tag{4.15}$$

and so

$$Sym\nabla I_3(\mathbf{A}) = \mathbf{A}^*, \tag{4.16}$$

where

$$\mathbf{A}^* = I_3\mathbf{A}^{-1} \tag{4.17}$$

if $\mathbf{A}$ is invertible.

5. Relations among gradients

In Elasticity we encounter the need to relate the gradients of the two sides of the equality $W(\mathbf{F}) = G(\mathbf{C})$ where $\mathbf{C} = \mathbf{F}^t\mathbf{F}$ and $\mathbf{F} \in Lin^+$ and $\mathbf{C} \in Sym^+$. A path $\mathbf{F}(t)$ in the former set induces a path $\mathbf{C}(t)$ in the latter and the equality may be differentiated to obtain

$$W_{\mathbf{F}} \cdot \dot{\mathbf{F}} = G_{\mathbf{C}} \cdot \dot{\mathbf{C}} = G_{\mathbf{C}} \cdot (\dot{\mathbf{F}}^t\mathbf{F} + \mathbf{F}^t\dot{\mathbf{F}}). \tag{5.1}$$

Using the symmetries of the inner product operation we write

$$G_C \cdot \dot{\mathbf{F}}'\mathbf{F} = \mathbf{F}(G_C)' \cdot \dot{\mathbf{F}} \quad \text{and} \quad G_C \cdot \mathbf{F}'\dot{\mathbf{F}} = \mathbf{F}(G_C) \cdot \dot{\mathbf{F}}, \tag{5.2}$$

so that

$$\{W_F - \mathbf{F}[G_C + (G_C)']\} \cdot \dot{\mathbf{F}} = 0. \tag{5.3}$$

Since the path is arbitrary, $\dot{\mathbf{F}}$ is an arbitrary element of *Lin*. The collection of terms in braces also belongs to *Lin*, therefore,

$$W_F = \mathbf{F}[G_C + (G_C)'] = 2\mathbf{F}(SymG_C). \tag{5.4}$$

The symmetry of $\mathbf{C}$ means that only the symmetric part of G_C is determinate and it is only this part which appears in the result. Indeed, we may use the fact that $\dot{\mathbf{C}}$ is symmetric to replace G_C by $SymG_C$ in eqn (5.1) at the outset.

If G is an isotropic function of $\mathbf{C}$ then it depends on the principal invariants $I_k(\mathbf{C})$ and the chain rule provides

$$\left\{(SymG_C) - \sum_{k=1}^{3} G_k[Sym(I_k)_C]\right\} \cdot \dot{\mathbf{C}} = 0, \tag{5.5}$$

where $G_k = \partial G/\partial I_k$. The term in braces, an element of *Sym*, is thus orthogonal to every other element of *Sym*. Therefore, it vanishes, yielding

$$SymG_C = (G_1 + I_1 G_2)\mathbf{I} - G_2\mathbf{C} + G_3\mathbf{C}^*. \tag{5.6}$$

6. Extensions

In the literature one often encounters component formulas like

$$\partial W/\partial F_{iA} = F_{iB}(\partial G/\partial C_{BA} + \partial G/\partial C_{AB}) \tag{6.1}$$

in place of eqn (5.4) above. However, we have seen that the representation of the gradient in terms of partial derivatives is possible only if the components of the tensor argument are all independent. This is not the case here because $C_{AB} = C_{BA}$ and so eqn (6.1) cannot be valid as it stands.

In practice, the issue is moot because the scalar-valued function G is usually given and a procedure like that demonstrated in the previous section may be used to compute the gradient. Nevertheless, eqn (6.1) arises frequently in theoretical studies and the question of its validity is thus of independent interest. First, we note that this formula follows immediately from the chain rule

$$\partial W/\partial F_{iA} = (\partial G/\partial C_{BC})\partial C_{BC}/\partial F_{iA}, \quad \text{with} \quad C_{BC} = F_{jB}F_{jC}, \tag{6.2}$$

provided that the partial derivatives $\partial G/\partial C_{BC}$ are interpreted in the usual sense of holding fixed all components other than the one with respect to which the derivative is taken. This suggests that we introduce an extension H of G from Sym^+ to Lin. Thus, $H(\mathbf{C})$ is defined for $\mathbf{C}$ in Lin and satisfies $H(\mathbf{C}) = G(\mathbf{C})$ for $\mathbf{C}$ in $Sym^+ \subset Lin$. We assume the extension to be differentiable in Lin and conclude, for any path $\mathbf{C}(t)$ in Sym^+, that

$$(H_{\mathbf{C}} - G_{\mathbf{C}}) \cdot \dot{\mathbf{C}} = 0 \tag{6.3}$$

for any $\dot{\mathbf{C}}$ in Sym. Because the first factor belongs to $Lin = Sym \oplus Skw$ it follows that

$$G_{\mathbf{C}} = H_{\mathbf{C}} + \mathbf{W}, \tag{6.4}$$

where $\mathbf{W} \in Skw$. Furthermore, since $H_{\mathbf{C}} = (\partial H/\partial C_{AB})\mathbf{E}_A \otimes \mathbf{E}_B$ it follows that eqn (6.1) holds with G replaced by H.

Given G, an obvious choice for H, which automatically satisfies the requirements of a smooth extension, is

$$H(\mathbf{C}) = G(Sym\mathbf{C}), \quad \mathbf{C} \in Lin. \tag{6.5}$$

Normally, this is the only extension discussed, either implicitly or explicitly. An exception is the paper by Cohen and Wang (1984), where the general issue is treated in detail. An obvious question arises as to whether or not any generality is lost if eqn (6.5) is adopted. To examine this, let H' be another extension of G to Lin; then, for $\mathbf{C} \in Sym^+$, $H'(\mathbf{C}) = H(\mathbf{C})$ identically and

$$(H'_{\mathbf{C}} - H_{\mathbf{C}}) \cdot \dot{\mathbf{C}} = 0 \tag{6.6}$$

for any $\dot{\mathbf{C}}$ in Sym. As before, we conclude that

$$H'_{\mathbf{C}} = H_{\mathbf{C}} + \mathbf{W}', \tag{6.7}$$

for some skew $\mathbf{W}'$, and it follows that *any* smooth extension of G may be used in eqn (6.1) without affecting the result. It follows that no generality is lost by adopting the obvious extension given by eqn (6.5).

REFERENCE

Cohen, H. and Wang, C.-C. (1984). A note on hyperelasticity. *Arch. Ration. Mech. Anal.* **85**, 213.

7. Korn's inequality

An easy calculation yields $2u_{(A,B)}u_{(A,B)} = u_{A,B}u_{A,B} + u_{A,B}u_{B,A}$, where round braces are used to denote symmetrization, i.e., $2u_{(A,B)} = u_{A,B} + u_{B,A}$. Write the second term as $(u_{A,B}u_B)_{,A} - u_{A,AB}u_B$, integrate over κ, and use the divergence theorem to obtain

$$2\int_{\kappa} u_{(A,B)}u_{(A,B)}dv = \int_{\kappa} |\nabla\mathbf{u}|^2 dv + \int_{\partial\kappa} \mathbf{N}\cdot[(\nabla\mathbf{u})\mathbf{u}]da - \int_{\kappa} \mathbf{u}\cdot\nabla(Div\mathbf{u})dv. \tag{7.1}$$

Write the third integrand on the right as $Div(\mathbf{u}Div\mathbf{u}) - (Div\mathbf{u})^2$ and apply the divergence theorem again. For the special case in which $\mathbf{u} = \mathbf{0}$ on $\partial\kappa$, all the boundary integrals vanish and we get

$$2\int_{\kappa} u_{(A,B)}u_{(A,B)}dv = \int_{\kappa} |\nabla\mathbf{u}|^2 dv + \int_{\kappa} (Div\mathbf{u})^2 dv. \tag{7.2}$$

This furnishes an example of *Korn's inequality*

$$k\int_{\kappa} u_{(A,B)}u_{(A,B)}dv \geq \int_{\kappa} |\nabla\mathbf{u}|^2 dv, \tag{7.3}$$

where k is a positive constant depending only on the shape of the region κ. In the present example ($\mathbf{u} = \mathbf{0}$ on $\partial\kappa$) we have $k = 2$ for all regions, and this is the optimum value because eqn (7.2) implies that eqn (7.3) is satisfied as an equality when $\mathbf{u}(\mathbf{x})$ has zero divergence, whereas strict inequality obtains in the general case. Furthermore, in this case the optimum Korn constant happens to be the same for all κ. If $\mathbf{u}$ vanishes on a portion of $\partial\kappa$, eqn (7.3) remains valid, but the constant k then depends on the region. The optimal constant is an eigenvalue of a variational problem associated with eqn (7.3).

Horgan's paper (1995) contains an accessible account of various applications of Korn's inequality to Mechanics.

REFERENCE

Horgan, C.O. (1995). Korn's inequalities and their applications in continuum mechanics. *SIAM Rev.* 4, 491–511.

8. Poincaré's inequality

Poincaré's inequality is the assertion that there is a positive constant c such that

$$\int_{\kappa} |\nabla\mathbf{u}|^2 dv \geq c\int_{\kappa} |\mathbf{u}|^2 dv. \tag{8.1}$$

As in the case of Korn's inequality this is most easily proved for the case in which $\mathbf{u} = \mathbf{0}$ on $\partial\kappa$. Let $f(x_A)$ be a single component of the vector field $\mathbf{u}(\mathbf{x})$. Then, f vanishes on the boundary.

Consider a cross section of κ defined by $x_3 = z(x_1, x_2)$, and let $z_m(x_1, x_2)$ be the minimum value of this function on $\partial\kappa$ for a given point (x_1, x_2) of the cross section. Then, z_m is the x_3-coordinate of a point on the boundary (draw a figure). For the fixed values of (x_1, x_2) in question we have $f(x_1, x_2, z_m) = 0$, and, therefore,

$$\begin{aligned} f(x_1, x_2, z) &= \int_{z_m}^{z} f_{,3}dx_3 = \int_{z_m}^{z} (\mathbf{E}_3 \cdot \nabla f)dx_3 \leq \int_{z_m}^{z} |\nabla f|\, dx_3 \\ &\leq \left(\int_{z_m}^{z} dx_3\right)^{1/2} \left(\int_{z_m}^{z} |\nabla f|^2\, dx_3\right)^{1/2} = \sqrt{z - z_m}\left(\int_{z_m}^{z} |\nabla f|^2\, dx_3\right)^{1/2} \\ &\leq \sqrt{z - z_m}\left(\int_{z_m}^{z_M} |\nabla f|^2\, dx_3\right)^{1/2}, \end{aligned} \tag{8.2}$$

where $z_M(x_1, x_2)$ is the maximum value of the function z over κ at the same values of (x_1, x_2) and we have made use of the Cauchy–Schwartz inequality in the 2nd line. We square and integrate with respect to x_3 to obtain

$$\int_{z_m}^{z_M} f^2 dx_3 \leq \frac{1}{2}h(x_1, x_2)^2 \int_{z_m}^{z_M} |\nabla f|^2\, dx_3 \leq \frac{1}{2}H^2 \int_{z_m}^{z_M} |\nabla f|^2\, dx_3, \tag{8.3}$$

where $h(x_1, x_2) = z_M - z_m$ and H is the maximum of h (the maximum thickness of κ in the x_3-direction). We now integrate eqn (8.3) over the x_1, x_2-plane to get

$$\int_\kappa f^2 dv \leq \frac{1}{2}H^2 \int_\kappa |\nabla f|^2\, dv. \tag{8.4}$$

Recalling that f is a component of $\mathbf{u}$, we apply eqn (8.4) three times to find that

$$\sum_{A=1}^{3} \int_\kappa |\nabla u_A|^2\, dv \geq c \int_\kappa u_A u_A dv, \tag{8.5}$$

where c is a positive constant and $\nabla u_A = u_{A,B}\mathbf{E}_B$. However, $\sum |\nabla u_A|^2 = \sum(\nabla u_A \cdot \nabla u_A) = u_{A,B}u_{A,B} = |\nabla \mathbf{u}|^2$. Thus, eqn (8.5) is just eqn (8.1).

Poincaré's inequality is a special case of the Sobolev inequalities. One can find a derivation for general $\mathbf{u}(\mathbf{x})$ in any book about Sobolev spaces (e.g. Sobolev 1963). The same inequality remains valid (with a different c, naturally) for functions $\mathbf{u}(\mathbf{x})$ that vanish on a part of $\partial\kappa$. See, for example, the book by Morrey (1966), which has had an enormous influence on the mathematical development of nonlinear elasticity theory.

REFERENCES

Morrey, C.B. (1966). *Multiple integrals in the calculus of variations*. Springer, Berlin.

Sobolev, S.L. (1963). *Applications of functional analysis in mathematical physics*. American Mathematical Society, Providence.

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Note: Figures are indicated by an italic *f* following the page number.

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